The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS | Autumn Semester 2010–11

MECHANICS | 2 hours

Attempt all THREE questions.

1 (i) Sketch the region R in the xy-plane bounded by the lines $y = \sqrt{x}$ and $y = x^3$.

Evaluate $\displaystyle\int\int_R 4xy - y^3 \; dydx$. ***(12 marks)***

(ii) The double integral I is given by

$$I = \iint_A x^2 y^2 e^{-(x^2+y^2)} dxdy,$$

where A is the interior of the circle $x^2 + y^2 = a^2$.

By changing to plane polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$I = \left(\int_0^{2\pi} \frac{1}{8}(1 - \cos 4\theta) d\theta\right)\left(\int_0^a r^5 e^{-r^2} dr\right).$$

Deduce that

$$I = \frac{\pi}{4}\int_0^a r^5 e^{-r^2} dr.$$

(13 marks)

2 (i) A particle of mass m moves in a plane with origin O, and its plane polar coordinates are $(r(t), \theta(t))$. It is subject to a force directed towards O of magnitude $mF(r)$. You are **given** that the radial and transverse components of Newton's Second Law reduce to:

$$\ddot{r} - r\dot{\theta}^2 = -F(r) \quad (1)$$
$$r^2\dot{\theta} = h \quad (2)$$

where h is a positive constant (and, in the usual notation, a dot over a variable denotes its time derivative). Make the substitution $u = r^{-1}$, and use (2) to show that

$$\dot{r} = -h\frac{du}{d\theta}$$

and deduce that (1) becomes

$$h^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^{-2}F(u^{-1}). \quad (3)$$

(11 marks)

(ii) For the case $F(r) = \mu r^{-3}$, where μ is a positive constant with $\mu > h^2$, show that

$$\frac{d^2u}{d\theta^2} - \alpha^2 u = 0,$$

where $\alpha^2 = \dfrac{\mu}{h^2} - 1$. Write down the general solution of this equation.

(7 marks)

Find the particular solution that satisfies $r = a$ and $\dot{r} = 0$ when $\theta = 0$.

(7 marks)

3 A light rod OAB of length $2a$ has particles of equal mass m fixed at A, where $OA = a$, and at B. It is free to rotate in a vertical plane about a smooth fixed axis through O.

(i) Show that the moment of inertia about the axis through O is $5ma^2$.

(4 marks)

(ii) The rod is released from rest with OB horizontal.

(a) Show that, when the rod has turned through an angle θ,

$$5a\dot{\theta}^2 = 6g\sin\theta.$$

(10 marks)

(b) Show that when the rod is vertical, the reaction at O is $\dfrac{28}{5}mg$.

(11 marks)

End of Question Paper