MAS204

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010-2011

NUMERICAL LINEAR ALGEBRA — Two hours

Answer FOUR questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1 (i) (a) Define the condition number, $\mathcal{K}(A)$, of a non-singular square matrix A for an arbitrary subordinate norm, $\|\cdot\|$. Show that $\mathcal{K}(A) \geq 1$.

(4 marks)

(b) Consider the perturbed matrix $A + \delta A$, for some small perturbation δA. It is possible that $A + \delta A$ is *singular.* By considering the equation

$$(A + \delta A)\mathsf{x} = 0$$

for nonsingular A, show that $A + \delta A$ *cannot* be singular if $\left\|A^{-1}\right\| \|\delta A\| < 1$.

(7 marks)

(ii) We wish to solve $A\mathsf{x} = \mathbf{b}$ where $\mathbf{b}$ is subject to an uncertainty $\delta\mathbf{b}$, and A is subject to an uncertainty δA. Thus, in effect we necessarily solve

$$(A + \delta A)(\mathsf{x} + \delta\mathsf{x}) = \mathbf{b} + \delta\mathbf{b}.$$

Assuming that $\left\|A^{-1}\right\| \|\delta A\| < 1$, then prove that the relative error in x satisfies

$$\frac{\|\delta\mathsf{x}\|}{\|\mathsf{x}\|} \leq \frac{\mathcal{K}(A)}{\left\{1 - \mathcal{K}(A)\frac{\|\delta A\|}{\|A\|}\right\}} \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}\right).$$

(9 marks)

(iii) Given

$$A \approx \begin{bmatrix} 1.0000 & 0.2500 & 0.1111 & 0.0625 \\ 0.2500 & 0.1111 & 0.0625 & 0.0400 \\ 0.1111 & 0.0625 & 0.0400 & 0.0278 \\ 0.0625 & 0.0400 & 0.0278 & 0.0204 \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} 7.0956 & -84.061 & 230.51 & -171.04 \\ -84.061 & 1397.1 & -4246.1 & 3304.5 \\ 230.51 & -4246.1 & 13684 & -11028 \\ -171.04 & 3304.5 & -11028 & 9121.8 \end{bmatrix},$$

$\|\delta A\|_\infty \approx 0.00002$, and $\|\delta\mathbf{b}\|_\infty = 0$, determine a bound on $\|\delta\mathsf{x}\|_\infty / \|\mathsf{x}\|_\infty$, and comment on the usefulness of solutions computed for this system.

(5 marks)

2 (i) Given $m+1$ data points $(x_j,\ f_j)$, $j = 0, 1, ..., m$, where the x_j values are all distinct, derive the normal equations for determining the least square fit to the data by the function

$$P_n(x) = \sum_{i=0}^{n} \alpha_i x^i$$

without weights.

(7 marks)

(ii) Hence, write down the normal equations for the best quadratic fit for the data

x_j	0.1000	0.2000	0.3000	0.4000
f_j	0.3528	1.6896	2.2264	1.8432 .

(8 marks)

(iii) Find the LU decomposition of the coefficient matrix of the normal equations, and solve the equations using the decomposition. Write down the expression of the quadratic fit of the data. ***(10 marks)***

3 (i) Verify that, if P is an orthogonal matrix, then $\|P\mathbf{x}\|_2 = \|\mathbf{x}\|_2$, where $\mathbf{x}$ is a column vector. ***(3 marks)***

(ii) The Householder reflection matrix is given by

$$P = \left(I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}} \right),$$

where $\mathbf{w}$ is a $(m+1)$ dimensional column vector. Show that P is orthogonal. ***(4 marks)***

(iii) Given the vector $\mathbf{x}^T = (x_0, x_1, ..., x_m)$ and the relation

$$\hat{\mathbf{x}} = P\mathbf{x},$$

where P, above, is based upon the vector

$$\mathbf{w} = \mathbf{x} + sign(x_0)\, ||\mathbf{x}||_2\, \mathbf{e}_0,$$

where $\mathbf{e}_0$ is the first column of the $(m+1) \times (m+1)$ identity matrix, show that

$$\hat{\mathbf{x}} = -sign(x_0)||\mathbf{x}||_2\, \mathbf{e}_0.$$

(6 marks)

(iv) Apply two Householder reflections to transform the matrix

$$A = \begin{bmatrix} 1 & 2.2 \\ 1 & 2.4 \\ 1 & 2.6 \\ 1 & 2.8 \end{bmatrix}$$

to the row echelon form. ***(12 marks)***

4 (i) The real symmetric matrix A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n| > 0$$

with corresponding linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ which can be supposed to be normalized so that the largest element of each one is unity.

(a) Write down the power iteration for finding the dominant eigenvalue and its eigenvector. ***(3 marks)***

(b) Prove that, given the assumptions above, the iterates converge to the dominant eigenvalue and its eigenvector. ***(11 marks)***

(ii) Show that if a matrix A has an eigenvalue λ, then the matrix $B \equiv A - pI$ has an eigenvalue $\lambda - p$. ***(2 marks)***

(iii) The matrix

$$A = \begin{pmatrix} 4.0 & 2.1 & 0.1 \\ 2.1 & 5.0 & 2.1 \\ 0.1 & 2.1 & -4.1 \end{pmatrix}$$

has eigenvalues given approximately by -4.6, 2.5 and 6.9 .

(a) Which of the three values 5, -5 or -7 is a suitable choice for p for the determination of the eigenvalue nearest to -4.6 using the power method with shift of origin? State the reason for your choice. ***(2 marks)***

(b) Use your chosen value of p together with $\mathbf{z}_0^T = (0, -0.2, 1.0)$ to compute **one** further estimate to the eigenvalue near -4.6 and its corresponding normalized eigenvector. Work correct to four decimal places, stating clearly your eigenvalue estimate. ***(4 marks)***

(iv) Show that, if A is a real symmetric matrix and has an eigenvalue λ_1 with corresponding eigenvector $\mathbf{x}_1$, then matrix $B = A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T$ has the eigenvalue 0 with corresponding eigenvector $\mathbf{x}_1$. You may assume $\mathbf{x}_1$ is normalized so that $\|\mathbf{x}_1\|_2 = 1$. ***(3 marks)***

5 (i) The linear system

$$A\mathbf{x} = \mathbf{b},$$

where A is an $n \times n$ matrix of known coefficients and $\mathbf{b}$ is an $n \times 1$ column vector of known values, can be rearranged in arbitrarily many ways in the form

$$\mathbf{x} = H\mathbf{x} + \mathbf{d}$$

which can subsequently be used to define the iteration

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{d}$$

where H is some $n \times n$ matrix and $\mathbf{d}$ is an $n \times 1$ column vector of known constant values.

(a) Show that

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}\| \leq \|H\|^{k+1} \; \|\mathbf{x}^{(0)} - \mathbf{x}\|,$$

and hence show that $\|H\| < 1$ is a sufficient condition for the iteration to converge to $\mathbf{x}$. ***(6 marks)***

(b) Starting from $A\mathbf{x} = \mathbf{b}$, derive the Jacobi iteration, and hence prove that **strict diagonal dominance** of the matrix A is sufficient to guarantee the convergence of the method. ***(7 marks)***

(ii) (a) Use the Gauss-Seidel iterative method to obtain **two** successive approximations to the solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 10 & 4 & 0 \\ 4 & 5 & 3 \\ 0 & 3 & 10 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

using $\mathbf{x}^{(0)} = (1, 1, 1)^T$ as the starting vector and working correct to four decimal places. ***(7 marks)***

(b) Show that, for the above system, the Gauss-Seidel iteration converges. ***(5 marks)***

End of Question Paper