Data provided: "Statistics Tables" by H.R. Neave

MAS205

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

Statistics Core **2 hours**

RESTRICTED OPEN BOOK EXAMINATION

Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
Candidates should attempt ***ALL*** *five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–21; Q2–22; Q3–19; Q4–10; Q5–28)

1 Let X be a random variable with probability density function $f_X(x)$ given by

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(-t \le X \le t)$ if $0 < t < 1$. ***(4 marks)***

(b) Let $Y = 2X + 3$. Find the probability density function of Y. ***(8 marks)***

(c) Let $Z = X^2$.

(i) Using your answer to (a), calculate the distribution function of Z. ***(5 marks)***

(ii) Hence find the probability density function of Z. ***(4 marks)***

2 Let $T \subseteq \mathbb{R}^2$ be the set $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, and let X and Y be random variables with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} 6(x-y)^2 & (x, y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \geq \frac{1}{2}, Y \leq \frac{1}{2})$. ***(7 marks)***

(b) Find the marginal probability density function of X. ***(5 marks)***

(c) Let $U = X+Y$ and $V = X-Y$. Find the joint probability density function of the random vector $(U, V)^T$, clearly stating the values for which it is non-zero. ***(10 marks)***

3 The random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ has a multivariate normal distribution. The variables X_1, X_2 and X_3 have means $2, -1$ and 3 respectively and variances $9, 25$ and 49 respectively. The variable X_1 is independent of each of X_2 and X_3, while X_2 and X_3 have correlation coefficient $-\frac{1}{5}$.

(a) Write down the mean vector of $\mathbf{X}$, and find its covariance matrix. ***(5 marks)***

(b) Let $Y_1 = 2X_1 + X_2 + X_3$ and $Y_2 = X_2 + kX_3$, for some real number k.

(i) If $\mathbf{Y} = (Y_1, Y_2)^T$, find the mean vector and covariance matrix of $\mathbf{Y}$, and hence give the distributions of Y_1 and Y_2 in the form $N(\mu, \sigma_2)$. ***(10 marks)***

(ii) For what value of k are Y_1 and Y_2 independent? ***(4 marks)***

4 An observation from a $Bi(3, \theta)$ distribution gives the value 2.

(a) Write down the likelihood of θ given this observation, and hence find the log likelihood. ***(4 marks)***

(b) Find the maximum likelihood estimate of θ given this observation. ***(6 marks)***

5 (a) Let $x_1, x_2, \ldots, x_n$ be a random sample from a $Be(1, \theta)$ distribution, with probability density function

$$f(x) = \begin{cases} \theta(1-x)^{\theta-1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases},$$

where $\theta > 0$ is unknown. Find the maximum likelihood estimate of θ. ***(16 marks)***

(b) The random variable X has a $Be(1, 3)$ distribution, and conditional on X, the random variable W has a Binomial distribution $Bi(4, X)$. Find the mean and variance of W. ***(12 marks)***

End of Question Paper