The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

Continuity and Integration **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Give the formal definition of the notion of a sequence of real numbers *converging to a limit.*

(a) Use the definition to show that the sequence $\dfrac{\sqrt{n}+1}{\sqrt{n}} \to 1$.

(b) Let (x_n) be a sequence with $x_n \to x$. Use the definition to show that $|x_n| \to |x|$. Give an example of a *divergent sequence* (x_n) such that the sequence of absolute values $(|x_n|)$ is convergent.

[**Hint:** You may use the inequality $|\,|a| - |b|\,| \leq |a-b|$ for all $a, b \in \mathbb{R}$ without proof.]

(c) Show that if (x_n) is a convergent sequence then we can find an $A > 0$ such that $|x_n| < A$ for all n.

(17 marks)

(ii) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief reasons in all cases, and state the limits if they exist.

$$\frac{20n^{11} + 2011n}{n(2n^5+1)^2}\ ; \qquad (-1)^n + \frac{1}{n}\ ; \qquad \left(2^{(1+1/n)} + 3 + \frac{1}{\sqrt{n+1}}\right)^2 .$$

(8 marks)

2 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided they are precisely stated.

(i) If a set of real numbers has a supremum then it has a maximum.

(ii) If a set of real numbers has a maximum then it has a supremum.

(iii) **Any** set of real numbers which is bounded below has an infimum.

(iv) It is possible for a set of real numbers to have a minimum which is also an upper bound for the set.

(v) If $x_n \to x, y_n \to y$ and $x_n \leq y_n$ then $x \leq y$.

(vi) If $x_n \to x, y_n \to y$ and $x_n < y_n$ then $x < y$.

(25 marks)

3 (i) Define what it means for a real-valued function f to be *continuous* at a point a in its domain.

For each of the following, give an example of a function with the stated property. You do not need to prove that your function has the required property.

(a) A continuous function $f : [0, 1) \longrightarrow \mathbb{R}$ which is not bounded below.

(b) A discontinuous function $f : [0, 2] \longrightarrow \mathbb{R}$ whose restriction to $[0, 1) \cup (1, 2]$ is continuous. It is sufficient to draw a graph.

(c) A bounded $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is discontinuous at every integer.

(d) A function $f : [0, 1] \longrightarrow \mathbb{R}$ which is bounded but has no maximum.

(12 marks)

(ii) State the Intermediate Value Theorem.

Show that the equation $x^2 + \cos^2 \dfrac{\pi x}{2} = \dfrac{7}{8}$ has two solutions in the interval $[0, 1]$.

[**Hint:** You may find $\cos \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ useful.] ***(13 marks)***

4 (i) Describe what it means for a function f to be *differentiable* at some point a interior to its domain. Show that if f is differentiable at a then it is continuous there.

For each of the following, give an example of a function with the stated property. You do not need to prove that your function has the required property.

(a) A continuous function $f : [0,1] \longrightarrow \mathbb{R}$ which is **not** differentiable at the points $\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$. It is sufficient to draw a graph.

(b) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is differentiable but not twice differentiable.

(13 marks)

(ii) State the Mean Value Theorem.

Now let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous on $[0,1]$, differentiable on $(0,1)$ and satisfy $f(1)^2 - f(0)^2 = 1$. Set $h(x) := f(x)^2 - x^2$.

(a) Indicate briefly why $h : [0,1] \longrightarrow \mathbb{R}$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and calculate its derivative there in terms of $f,\ f'$ and x.

(b) Apply the Mean Value Theorem to h and deduce that there is a $c \in (a,b)$ such that $f'(c)f(c) = c$. ***(12 marks)***

5 (i) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function.

(a) Explain what it means to say that f is *Riemann integrable.*

(b) For a partition of $[a,b]$, describe what is meant by its *modulus* and what is meant by a *Riemann sum* corresponding to the given partition.

(c) State Darboux' Theorem. ***(11 marks)***

(ii) Provide, with proof, an example of a bounded function $f : [0,1] \longrightarrow \mathbb{R}$ which is not Riemann integrable. ***(6 marks)***

(iii) Evaluate the limit

$$\lim_{n\to\infty} \frac{1^{2010} + 2^{2010} + 3^{2010} + \cdots + n^{2010}}{n^{2011}}.$$

You may assume the Fundamental Theorem of Calculus, but you must indicate where and how you have used it. ***(8 marks)***

End of Question Paper