The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

Topics in Number Theory (Level 2) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at your answers.

1 (i) You publish $(n, e) = (133, 13)$ in the RSA directory and receive 15. Decode it. ***(12 marks)***

(ii) State *Wilson's Theorem* and determine the remainder when 26! is divided by

(a) 27,

(b) 29,

(c) 87. ***(10 marks)***

(iii) For a prime number $p > 2$, determine the remainder when $2^{p-1} + (p-1)!$ is divided by p. ***(3 marks)***

2 (i) State the *Law of Quadratic Reciprocity.* ***(2 marks)***

(ii) Calculate the Legendre symbol $\left(\frac{76}{103}\right)$ and deduce whether the congruence

$$x^2 + 12x + 17 \equiv 0 \pmod{103}$$

has a solution. If it has, solve it. ***(9 marks)***

(iii) Use the Law of Quadratic Reciprocity to prove that, for a prime number $p > 3$,

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12}. \end{cases}$$

(8 marks)

Deduce that, if $p > 3$ is a prime divisor of $m^2 - 3n^2$, where m, n are coprime positive integers, then $p \equiv 1$ or $11 \pmod{12}$. Illustrate this in the case $(m, n) = (17, 3)$. ***(6 marks)***

3 (i) State *Euler's Criterion.* ***(1 mark)***

For each of the numbers

$$2^{81} - 1, \quad 2^{82} + 1, \quad 2^{83} - 1, \quad 2^{84} + 1,$$

find a prime number which divides it. ***(10 marks)***

(ii) Let $\sigma(n)$ denote the sum of the positive divisors of the natural number n. Calculate $\sigma(1184)$ and $\sigma(1210)$ and show how, for the numbers 1184 and 1210, the sum of the proper positive divisors of each one is related to the other. ***(9 marks)***

(iii) Define a *perfect number* and show that a power of a prime number cannot be perfect. ***(5 marks)***

4 State the formulae which give all primitive Pythagorean triples. ***(2 marks)***

(i) Determine all primitive Pythagorean triples in which one of the numbers is 208. ***(6 marks)***

(ii) Give a non-primitive Pythagorean triple in which one of the numbers is 208. ***(2 marks)***

(iii) Determine all primitive Pythagorean triples in which one of the numbers is 305. ***(13 marks)***

(iv) Give a non-primitive Pythagorean triple in which one of the numbers is 305. ***(2 marks)***

5 (i) Express $\sqrt{2}$ as a continued fraction, find a convergent of $\sqrt{2}$ which differs from it by less than 10^{-4} and find three solutions of the Pell equation $x^2 - 2y^2 = 1$ in positive integers. ***(11 marks)***

(ii) State *Binet's Formula* for the n-th Fibonacci number f_n. ***(1 mark)***

Let p be a prime number. Show that

$$f_p \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv 1 \text{ or } 4 \pmod{5} \\ -1 \pmod{p} & \text{if } p \equiv 2 \text{ or } 3 \pmod{5}. \end{cases}$$

Illustrate this result with f_7 and f_{11}, and comment on the case $p = 5$.

(13 marks)

End of Question Paper