Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find and classify the stationary points of the function

$$g(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1.$$

(15 marks)

(ii) Demonstrate that the function

$$f(x) = e^x - x^2 - 4.0$$

has a root between $x = 2.0$ and $x = 2.2$. Write down the iteration formula for the Newton-Raphson method. Starting with $x_0 = 2.0$, use three iterations of the Newton-Raphson method to estimate this root. Work correct to five decimal places throughout. ***(10 marks)***

2 A periodic function $f(t)$ of period 2 is defined by

$$f(t) = \begin{cases} 0 \text{ for } & -1 \le t < 0 \\ 1 \text{ for } & 0 \le t < 1 \end{cases}$$

$$f(t + 2) = f(t).$$

Show that the first four non-zero terms of the Fourier series expansion of $f(t)$ are given by

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + ...\right).$$

(19 marks)

Sketch a graph of $f(t)$ for $-6 \le t \le 6$. ***(6 marks)***

3 (i) Show that the directional derivative of the function

$$\Phi = x^2y^2 + xyz + ye^z$$

in the direction of the vector $\mathbf{v} = (1, 3, -1)$ at the point $(1, 2, 1)$ is $\dfrac{(23 + e)}{\sqrt{11}}$.

(9 marks)

(ii) For the vector

$$\mathbf{a} = (6x + By + Cxz, 2x + 3y + Fz, 4x^2 + 6y + z),$$

find the constants B, C and F such that $\nabla \times \mathbf{a} = \mathbf{0}$ for all x, y, z. For these values of B, C and F, find a scalar potential ψ such that

$$\mathbf{a} = \nabla\psi.$$

(16 marks)

4 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} + 2\frac{\partial^2 y}{\partial x \partial t} - 3\frac{\partial^2 y}{\partial x^2} = 0,$$

has solutions of the form $y = f(x + \lambda t)$ for arbitrary functions f provided that $\lambda = -3$ or $\lambda = 1$.

(8 marks)

Give an interpretation, including a clear diagram, of the form of the solution in each case.

(6 marks)

Derive the solution that satisfies the conditions

$$y(x, 0) = 0$$

$$\frac{\partial y}{\partial t}(x, 0) = 3x.$$

(11 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$