



The  
University  
Of  
Sheffield.

MAS253

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2010–11

MATHEMATICS FOR ENGINEERING MODELLING

2 hours

*Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.*

- 1 (i) Derive the Maclaurin series for  $\sinh(x)$  up to the  $x^5$  term and for  $\cosh(x)$  up to the  $x^4$  term. Use the series to show that, up to and including terms in  $x^5$ ,

$$2 \sinh(x) \cosh(x) = \sinh(2x).$$

*(10 marks)*

- (ii) Evaluate

$$\lim_{x \rightarrow \infty} (x^2 e^{-2x}).$$

*(5 marks)*

- (iii) Derive the partial sum  $s_n$  to  $n$  terms of the infinite series

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots \quad (r \neq 1).$$

Hence show that the series is convergent for  $|r| < 1$  and find the sum.

Find the sum of the infinite geometric series

$$2 + 6x + 18x^2 + 54x^3 + \dots + 2(3x)^n + \dots, \quad (1)$$

and state the value of the radius of convergence,  $R$ , of the series. Hence find, for  $|x| < R$ , the sum of the infinite series

$$6 + 36x + 162x^2 + \dots + 2n(3x)^{n-1} + \dots \quad (2)$$

*(10 marks)*

- 2 (i) Denote the Fourier series of

$$f(x) = \begin{cases} -x, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

by  $F(x)$ , where  $F(x)$  has a period  $2\pi$ . Show that

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos(2m-1)x}{(2m-1)^2}.$$

*(16 marks)*

- (ii) For  $-2\pi < x < 2\pi$  sketch the graph of the sum of the series.

*(4 marks)*

- (iii) Deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

*(5 marks)*

- 3 (i) Find *by integration* the Laplace transform  $F(s)$  of  $t$ . (You may assume that  $s > 0$ .) State the Shift Theorem and use it to deduce the Laplace transform of  $te^{2t}$ .

(3 marks)

- (ii) Verify that

$$\frac{1}{s^2(s^2 - 3s + 2)} = \frac{1}{4} \left\{ \frac{1}{s-2} - \frac{4}{s-1} + \frac{3}{s} + \frac{2}{s^2} \right\}.$$

(4 marks)

- (iii) Use the method of Laplace transforms to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = t,$$

subject to the initial conditions  $y(0) = 0$ ,  $dy/dt = 0$  at  $t = 0$ .

(9 marks)

- (iv) A particle is attached to one end of a light spring, whose other end is fixed. The particle moves in a vertical line and its equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2x = V\delta(t - T),$$

where  $x$  is the distance below its equilibrium position,  $\delta(t - T)$  is the Dirac delta function and  $V$  is a constant. Find  $x(t)$  using Laplace transforms, given that  $x(0) = 0$ ,  $dx/dt = V/2$  at  $t = 0$ .

Sketch  $x\omega/V$  against  $\omega t$  for  $0 \leq \omega t \leq 3\pi$  for the case  $\omega T = \pi$ . Give a very brief physical description of the motion and your solution.

(9 marks)

- 4 In a large slab of thickness  $l$  the temperature  $\varphi(x, t)$  at a distance  $x$  from one face satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial t} = \kappa \frac{\partial^2 \varphi}{\partial x^2},$$

where  $\kappa$  is a positive constant. The faces  $x = 0$  and  $x = l$  are both insulated (so that  $\partial \varphi / \partial x = 0$  on the faces) and at  $t = 0$  the temperature for  $0 < x < l$  is  $\varphi_0 x / l$ , where  $\varphi_0$  is a constant.

- (i) Use the *method of separation of variables* to show that  $\varphi$  takes the form

$$\varphi(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\kappa n^2 \pi^2 t / l^2}.$$

(16 marks)

- (ii) Determine the coefficients  $A_0$  and  $A_n$  ( $n = 1, 2, \dots$ ).

(7 marks)

- (iii) Write down the value of  $\lim_{t \rightarrow \infty} \varphi(x, t)$  and give a brief physical interpretation.

(2 marks)

- 5 (i) (i) Evaluate

$$\int_{x=1}^{x=4} \int_{y=-1}^{y=2} (2x + 6x^2y) dy dx.$$

(5 marks)

- (ii) Use polar coordinates to evaluate

$$\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} dy dx.$$

(8 marks)

- (iii) (iii) Sketch the region of integration of the double integral

$$I = \int_{y=0}^1 \int_{x=y}^1 (1-x^2)^4 dx dy.$$

By changing the order of integration evaluate  $I$ .

(12 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2 \sin P \cos P ,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2 \cos^2 P - 1 = 1 - 2 \sin^2 P ,$$

$$\cos P \cos Q = \frac{1}{2} \{ \cos (P + Q) + \cos (P - Q) \} ,$$

$$\sin P \sin Q = -\frac{1}{2} \{ \cos (P + Q) - \cos (P - Q) \} ,$$

$$\sin P \cos Q = \frac{1}{2} \{ \sin (P + Q) + \sin (P - Q) \} .$$

Geometric progression

The partial sum to  $n$  terms of

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots$$

is

$$S_n = a(1 - r^n)/(1 - r), \quad r \neq 1 .$$

Taylor Series for functions of one variable (x)

The Taylor series of  $f(x)$  about  $x=a$  is

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \end{aligned}$$

The Maclaurin series of  $f(x)$  is the special case of the Taylor series when  $a=0$ :

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots \quad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \quad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \quad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \quad (R=1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots \quad (R=1)$$

$R$  is the radius of convergence.

### Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

If  $n$  is positive and integer, series terminates.

If  $n$  is negative or non-integer (or both), the series is an infinite series with the radius of convergence,  $R=1$ .

### Fourier Series

The Fourier series of  $f(x)$  for  $-l < x < l$  is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx \quad ,$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \quad n=1,2,\dots$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \quad n=1,2,\dots$$

### Laplace Transform

The Laplace Transform of  $f(t)$  is

$$F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad .$$

For special cases, see later page.

## Partial Differentiation

$$\delta F = F(x + \delta, y + \varepsilon) - F(x, y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that  $F = F(x, y)$  and that  $x$  and  $y$  are functions of  $t$ , i.e.  $x = x(t), y = y(t)$ , then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} .$$

2. Suppose that  $F = F(x, y)$  and that  $x$  and  $y$  are functions of the variables  $u$  and  $v$ , i.e.  $x = x(u, v), y = y(u, v)$ , then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial u}; \quad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial v} .$$

## Taylor Series for functions of two variables (x, y)

The Taylor series of  $f(x, y)$  about  $x = a, y = b$  is

$$\begin{aligned} f(x, y) &= f(a, b) + \{(x - a) f_x(a, b) + (y - b) f_y(a, b)\} + \\ &+ \frac{1}{2!} \{(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + \\ &+ (y - b)^2 f_{yy}(a, b)\} + \\ &+ \dots \end{aligned}$$

Here  $f_x = \frac{\partial f}{\partial x}$  etc.

## Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{Laplace's equation}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K} \frac{\partial V}{\partial t} \quad \text{Heat conduction (or diffusion) eqn.}$$

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} \quad \text{Wave equation}$$

## General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A \cos \omega x + B \sin \omega x$$

$$X'' = \omega^2 X \Rightarrow X(x) = C \cosh \omega x + D \sinh \omega x$$

$$\text{or } E e^{\omega x} + F e^{-\omega x}$$

$$T' = kT \Rightarrow T(t) = A e^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
$t$	$1/s^2$
$t^{n-1}/(n-1)! (n \geq 1)$	$1/s^n$
$e^{at}$	$\frac{1}{s-a}$
$\frac{1}{a} \sin at$	$\frac{1}{s^2 + a^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$
$\frac{1}{a} \sinh at$	$\frac{1}{s^2 - a^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$\frac{\sin at - at \cos at}{2a^3}$	$\frac{1}{(s^2 + a^2)^2}$
$\frac{t \sin at}{2a}$	$\frac{s}{(s^2 + a^2)^2}$
$e^{at} f(t)$	$F(s-a)$ , where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	$e^{-as}$
$u(t-a)$	$e^{-as}/s$
$u(t-a)f(t-a)$	$e^{-as} F(s)$ , where $F(s) = L(f(t))$