The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2010-2011**

Vectors and Fluids — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Given the vector field

$$\mathbf{F} = (2zx - yz,\, y^2 - xz,\, x^2 - xy),$$

calculate

$$\boldsymbol{\nabla} \cdot \mathbf{F}, \quad \boldsymbol{\nabla} \times \mathbf{F} \quad \text{and} \quad \boldsymbol{\nabla}^2 \mathbf{F}.$$

(6 marks)

Deduce that there is a scalar field $\phi = \phi(x, y, z)$ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$, and find $\phi(x, y, z)$.

(6 marks)

Verify that your answers are consistent with the result

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \boldsymbol{\nabla}^2 \mathbf{F}.$$

(2 marks)

(ii) Let

$$\phi = \frac{1}{r}, \qquad \text{where} \qquad r = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{and} \qquad \frac{\partial \phi}{\partial x} = -\frac{x}{r^3}.$$

(5 marks)

Hence show that ϕ satisfies the Laplace Equation.

(6 marks)

2 (i) Show that $\epsilon_{1jk}\, a_j\, b_k$ is equal to $(\mathbf{a} \times \mathbf{b})_1$.

(4 marks)

Use suffix notation to show that $\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v})$ can be written in terms of $\mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$ and $\mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u})$.

(9 marks)

(ii) Show that

$$L = \frac{1}{11} \begin{pmatrix} -7 & 6 & 6 \\ -6 & -9 & 2 \\ 6 & -2 & 9 \end{pmatrix}$$

is a valid matrix of transformation, representing a rotation of frames about a common origin.

(8 marks)

Find α such that the vector $\lambda = (1, 0, \alpha)$ points along the axis of rotation between the frames.

(4 marks)

3 Consider the curvilinear coordinate system given by

$$x = \cosh\eta\, \cos\psi, \qquad y = \sinh\eta\, \sin\psi, \qquad z = \gamma.$$

Find expressions for δx, δy and δz in terms of $\delta\eta$, $\delta\psi$ and $\delta\gamma$.

(7 marks)

Use your results to find an expression for

$$\delta l^2 = \delta x^2 + \delta y^2 + \delta z^2,$$

in terms of $\delta\eta$, $\delta\psi$ and $\delta\gamma$, and state why the coordinates (η, ψ, γ) must be orthogonal. Find expressions for h_1, h_2 and h_3 (in standard notation) in terms of η, ψ and γ.

(8 marks)

Consider the coordinate surface $z = \gamma = 0$, i.e. the (x, y)-plane. Within the (x, y)-plane, show that the coordinate curves for $\eta = \mathit{const}$ correspond to a family of ellipses. Given that

$$\sinh\eta \to 0, \quad \cosh\eta \to 1 \quad \text{as} \quad \eta \to 0, \qquad \text{and} \qquad \sinh 1.4 \approx \cosh 1.4 \approx 2,$$

sketch coordinate curves for $\eta = \mathit{const}$ in the range $(0, 1.4]$. Hence, or otherwise, sketch coordinate curves for $\psi = \mathit{const}$.

(10 marks)

4 Consider the steady two-dimensional flow

$$\mathbf{u}(x,y) = \frac{U}{L^2}\left\{(Lx + x^2 - y^2)\mathbf{i} - (Ly + 2xy)\mathbf{j}\right\},$$

where U and L are scalar constants.

(i) Show that the flow $\mathbf{u}$ is both incompressible and irrotational.

(5 marks)

(ii) Find the coordinates of all stagnation points.

(6 marks)

(iii) Find a velocity potential, ϕ such that $\mathbf{u} = \boldsymbol{\nabla}\phi$, with $\phi = 0$ at $x = y = 0$.

(5 marks)

(iv) Find a stream function, ψ such that $\mathbf{u} = \boldsymbol{\nabla}\times(\psi\mathbf{k})$, with $\psi = 0$ at $x = y = 0$.

(5 marks)

(v) Verify that

$$\phi + \mathrm{i}\,\psi = \frac{U}{6L^2}(2z^3 + 3Lz^2)\,,$$

where $\mathrm{i} = \sqrt{-1}$ and $z = x + \mathrm{i}\,y$.

(4 marks)

5 A sphere of radius a, centred on the origin, is fixed within an incompressible flow. For spherical polar coordinates,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

with $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$ in the standard notation. Given the velocity potential,

$$V = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta,$$

find the velocity field $\mathbf{u} = \boldsymbol{\nabla}V$. Verify that the appropriate boundary conditions are satisfied at $r = a$, and that the flow far from the sphere is $U\mathbf{k}$.

(11 marks)

Use Bernoulli's integral to find the pressure p, with $p \to p_0$ as $r \to \infty$.

(7 marks)

Explain why F, the z-component of the force exerted by the fluid on the sphere, is

$$-\int_S p\cos\theta\,\mathrm{d}S,$$

where S is the surface of the sphere. Performing the integral it is found that $F = 0$. Comment briefly on this result.

(7 marks)

End of Question Paper