The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

MAS271 Methods for differential Equations **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Solve the equation

$$(1-x^2)\frac{d^2y}{dx^2} \; - \; 5x\frac{dy}{dx} - 4y = 0$$

near the ordinary point $x = 0$. Give your solution, specifically including the first four even powers of x and the first four odd powers of x. ***(14 marks)***

(ii) Using appropriate Liapunov function of the form

$$V(x,y) = \alpha x^{2m} \; + \; \beta y^{2n}$$

where the constants α, β, m, n are to be appropriately chosen, show that the origin is asymptotically stable for the following system

$$\dot{x} = -x^3 + y\sin x$$
$$\dot{y} = -y - x^2 y - x\sin x$$

(5 marks)

(iii) Consider the system

$$\dot{x} = -y^3 + 2x^3$$
$$\dot{y} = x^3 + 3y^3$$

Try the Liapunov function $V(x,y) = ax^4 \; + \; by^4$ with a, $b > 0$ constants to be determined. Show using Liapunov's theorem if (0,0) is stable or not. ***(6 marks)***

2 Find the equilibrium points for the following equation

$$\ddot{\theta} = \sin\theta(\cos\theta - \alpha^2) \text{ for } \alpha^2 > 1 \text{ and } \alpha^2 = 1/2.$$

(7 marks)

What can you conclude about the nature of the equilibrium points. ***(13 marks)***

Sketch the trajectories in the xy-plane, indicating the direction of time, t, around the stable equilibrium point for the case: $\alpha^2 = 1/2$. ***(5 marks)***

3 (i) Find the general solution of

$$\text{(i)} \;\; y'' + 3py' + 2p^2y = 0$$
$$\text{(ii)} \;\; y'' + 2py' + p^2y = 0$$

for a constant p. ***(6 marks)***

(ii) Are the following functions positive definite, negative definite, or neither?

$$(a) \;\; x^2 - xy - y^2$$

$$(b) \;\; 2x^2 - 6xy + 6y^2$$

(8 marks)

(iii) Find the normal form of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

(11 marks)

4 (i) Show that $y = 3 - 12x^2 + 4x^4$ is a solution of

$$y'' - 2xy' + 8y = 0$$

(2 marks)

(ii) Find the value of constant λ for which the following problem has a non-trivial solution and, for each value of λ, find the non-trivial solution.

$$(xy')' + \frac{\lambda}{x}y = 0, \qquad y(1) = y(\mathrm{e}^{\pi}) = 0.$$

[Hint: Note that $x^{\pm i\mu} = \exp[\pm i\mu \ln x] = \cos[\mu \ln x] \pm i \sin[\mu \ln x]$.]
(15 marks)

Question 4 continued on next page

4 (continued)

(iii) By means of the substitution $t = \ln x$, show that Bessel's equation of the order ν is transformed to the normal form

$$\frac{d^2y}{dt^2} + (\mathrm{e}^{2t} - \nu^2)y = 0$$

(8 marks)

5 (i) Find one of the two linearly independent solutions of

$$y'' + \frac{1}{4x^2}y = 0$$

about $x = 0$, using Frobenius series method. ***(9 marks)***

(ii) Find the two equilibrium points of

$$\dot{x} = y(x+1), \quad \dot{y} = x(y^3+1)$$

and investigate the nature of the stability of the system at these points. Show the signs of $\dot{x}$ and $\dot{y}$ in all regions near the equilibrium points in the x-y plane. ***(16 marks)***

End of Question Paper