Data provided: "Statistics Tables" by H.R. Neave

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Probability Modelling **2 hours**

RESTRICTED OPEN BOOK EXAMINATION

Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–16; Q2–18; Q3–21; Q4–25; Q5–20)

1 A (delayed) renewal process is defined by tossing a fair coin repeatedly, and saying that a renewal occurs whenever a run of three consecutive heads is completed.

(a) Let f_n be the probability that, given that a renewal occurs at time t, the next renewal occurs at time $t + n$. Find the values of f_1 and f_2.
(4 marks)

(b) Find the expected number of tosses until the first renewal. ***(12 marks)***

2 A discrete time Markov chain has state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix given by

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Classify the states, and identify which classes are closed and which are not.
(8 marks)

(b) Find all stationary distributions of the chain. ***(10 marks)***

3 Summer days at a particular location can be classified as "hot and dry" (state 1), "thundery" (state 2) or "cool and wet" (state 3). Assume that this can be modelled as a Markov chain on three states, $\{1, 2, 3\}$ with transition matrix given by

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

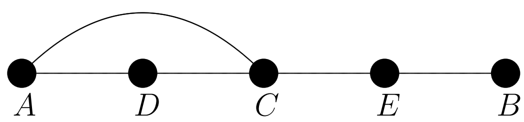
(a) Show that the chain is irreducible and aperiodic. ***(6 marks)***

(b) Find the stationary distribution of the chain. ***(7 marks)***

(c) If today is known to be thundery,

(i) find the probability that the day after tomorrow is cool and wet; ***(4 marks)***

(ii) give an approximate value for the probability that a day in two months' time is cool and wet (assuming the model holds until then). ***(4 marks)***



4 Consider a particle performing a random walk on the graph shown in the figure above.

(a) Write down the transition matrix of the random walk considered as a Markov chain on five states $\{A, B, C, D, E\}$. ***(5 marks)***

(b) Given that the particle starts at vertex C, find the probability that it reaches vertex A before vertex B. ***(10 marks)***

(c) Given that the particle starts in vertex C, find the expected time until it reaches either vertex A or vertex B. ***(10 marks)***

5 Customers arrive in shop A as a Poisson process with rate $\lambda_A = 4$, for $t \geq 0$, and in shop B as a variable rate Poisson process with rate $\lambda_B(t) = 2t$, for $t \geq 0$. In both shops each customer, independently of their arrival time and of other customers, spends a quantity which is a positive random variable (measured in pounds) with mean 12 and variance 25.

(a) Find the probability that exactly one customer arrives at shop B before time 1. ***(5 marks)***

(b) Find the mean and variance of the total amount spent in shop A by customers who arrive before time 9. ***(6 marks)***

(c) Give an approximate probability that the total amount spent in shop A by customers who arrive before time 9 is at least £400. ***(9 marks)***

End of Question Paper