The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Vector spaces and Fourier theory **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Let V be a vector space, and let $\mathcal{V}$ denote the collection of vectors $v_1, \dots, v_n$ in V. Define what it means for $\mathcal{V}$ to

(a) be linearly independent;

(b) span V;

(c) be a basis for V. ***(6 marks)***

(ii) For each of the following subsets of $\mathbb{R}^3$, say whether it is a subspace, justifying your answer.

(a) $U_1 = \{(1, 2, 3)^T\}$;

(b) $U_2 = \{(x, y, z)^T \in \mathbb{R}^3 \mid y = 3\}$;

(c) $U_3 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x \leq z\}$;

(d) $U_4 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x + 3y + z = 0\}$. ***(7 marks)***

(iii) If U and W are subspaces of a vector space V, define $U + W$ and show that it is a subspace of V. ***(7 marks)***

(iv) In the vector space $V = \mathbb{R}^3$, let U be the subspace $\{(a, 2a, a)^T \mid a \in \mathbb{R}\}$, and W be the subspace $\{(x, y, z)^T \mid x = y\}$. Prove that $U \cap W = \{0_V\}$, and $V = U + W$. ***(5 marks)***

2 (i) State the Steinitz Exchange Lemma. ***(2 marks)***

(ii) Let $\mathcal{V} = \{v_1, \ldots, v_m\}$ be a set of vectors in an n-dimensional vector space V. Say whether or not each of the following statements is necessarily true, and, if not, give a counterexample:

(a) If $\mathcal{V}$ is linearly independent, then $m \leq n$;

(b) If $m \leq n$, then $\mathcal{V}$ is linearly independent;

(c) If $\mathcal{V}$ is a basis, then $m = n$;

(d) If $m = n$, then $\mathcal{V}$ is a basis. ***(6 marks)***

(iii) Let V and W be vector spaces and let $\phi : V \to W$ be a linear map. Define the kernel of ϕ and prove that ϕ is injective if and only if $\ker(\phi) = \{0_V\}$. ***(7 marks)***

(iv) Let V be a finite dimensional vector space and let U and W be subspaces of V. Give an equation relating the dimensions of $U \cap W$, U, W and $U + W$. ***(1 mark)***

(v) For $V = M_3(\mathbb{R})$ and $\mathbf{v} = (1, 0, 0)^T$, let

$$U = \{A \in V \mid A^T = -A\},$$
$$W = \{A \in V \mid A\mathbf{v} = 0\}.$$

You may assume that U and W are subspaces of V.

(a) Find the dimensions of U, W, $U \cap W$ and $U + W$. ***(8 marks)***

(b) Write down an element of V which is not in $U + W$. ***(1 mark)***

3 Consider the vector space $V = \mathbb{R}[x]_{\leq 3}$ of real polynomials in the variable x of degree at most 3.

(i) Explain what the standard basis of the vector space V is and state the dimension of V. ***(3 marks)***

(ii) Show that the list of polynomials

$$1, x - 1, x^2 - x, x^3 - x^2$$

is another basis for V. ***(4 marks)***

Consider the function $\phi : V \to V$ given by $\phi(f) = (1 + x)f'(x)$.

(iii) Show that ϕ is a linear map. ***(3 marks)***

(iv) Find the matrix of ϕ with respect to the standard basis. ***(3 marks)***

(v) Find the matrix of ϕ with respect to the basis in part (ii). ***(5 marks)***

(vi) What is the kernel of ϕ ? ***(2 marks)***

(vii) Show that the image of ϕ is $\{f \in V \mid f(-1) = 0\}$. ***(5 marks)***

4 (i) Let $V = \mathbb{R}[x]$, the vector space of real polynomials in the variable x, with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Let U be the subspace of V of spanned by x, x^2, x^3.

(a) Apply the Gram-Schmidt process to produce an orthogonal basis for U. ***(6 marks)***

(b) Calculate the closest element of U to the constant polynomial 1. ***(6 marks)***

(c) What is the distance from this element to 1? ***(5 marks)***

(ii) Let V be an inner product space (over $\mathbb{R}$).

(a) State the Cauchy-Schwarz inequality for vectors $v, w \in V$, including the criterion for when equality holds. ***(3 marks)***

(b) By considering $C[0,\pi]$ with the inner product

$$\langle f, g\rangle = \int_{0}^{\pi} f(t)g(t)\,dt,$$

show that

$$\int_0^{\pi} \sqrt{\sin t}\,dt \le \sqrt{2\pi}.$$

(5 marks)

5 (i) Explain what the standard inner product on $M_2(\mathbb{R})$ is and calculate the distance from $A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ to $B = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$. ***(5 marks)***

(ii) Consider the Fourier inner product space $C[-\pi,\pi]$ of continuous functions $[-\pi,\pi] \to \mathbb{R}$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

(a) Is the set $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$ orthogonal? Is it orthonormal? Justify your answers briefly. ***(2 marks)***

(b) Calculate the angle between $\cos 2t \cos t$ and $\cos 4t \cos t$. ***(8 marks)***

(iii) Let V be the subspace of $C[-\pi,\pi]$ spanned by the set $\mathcal{V} = \{1, \cos 3t, \sin 3t\}$. Consider the linear map $D : V \to V$ given by differentiation, $D(f) = f'$.

(a) What is the matrix of D with respect to the basis $\mathcal{V}$? ***(3 marks)***

(b) If $\widehat{D}$ denotes the adjoint of $D : V \to V$, show that $\widehat{D} = -D$. ***(7 marks)***

End of Question Paper