The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Summer 2010–2011**

Nonlinear Mathematics **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Let $\varphi\colon A \to B$ be a map where A and B are subsets of $\mathbb{R}^2$.

(a) For $(u,v) \in B$ define the preimage $\varphi^{-1}(u,v)$ of (u,v) under φ.

(b) For $T \subseteq B$ define the preimage $\varphi^{-1}(T)$ of T under φ.

(3 marks)

(ii) Define $A = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$ and consider the map $\varphi\colon A \to \mathbb{R}^2$ defined by $\varphi(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Write $T = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 = 4\}$. Find the preimage $\varphi^{-1}(T)$. **(3 marks)**

(iii) (a) Let A and B be any subsets of $\mathbb{R}^2$ and let $\varphi\colon A \to B$ be a bijective map. Define the inverse map $\varphi^{-1}\colon B \to A$.

(b) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a function and define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x,y) = (F(x,y),\, y)$.

Suppose that φ is a bijection. Show that its inverse $\varphi^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ has the form $\varphi^{-1}(u,v) = (G(u,v), v)$ for some function $G\colon \mathbb{R}^2 \to \mathbb{R}$.

Show that there exists a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $F(g(y),y) = 0$ for all $y \in \mathbb{R}$.

(8 marks)

(iv) Define $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$ by $\varphi(x,y,z,t) = (u,v,w)$, where

$$u = 2(xt - yz), \quad v = 2(xy + zt), \quad w = x^2 - y^2 + z^2 - t^2.$$

Find $\varphi^{-1}(0,1,0)$. **(11 marks)**

2 (i) Let $(a,b) \in \mathbb{R}^2$ and let $r > 0$. Define the *open ball* $B((a,b),r)$.

Define what it means for a set $A \subseteq \mathbb{R}^2$ to be *open*. **(4 marks)**

(ii) (a) Fix a point $(a,b) \in \mathbb{R}^2$ and write $A = \mathbb{R}^2 \backslash \{(a,b)\}$. Prove that A is an open set. **(4 marks)**

(b) Let A_1 and A_2 be open sets in $\mathbb{R}^2$. Prove that the intersection $A_1 \cap A_2$ is an open set in $\mathbb{R}^2$. **(4 marks)**

(c) Let $A \subseteq \mathbb{R}^2$ be an open set and let $(a,b) \in A$. Prove that $A \backslash \{(a,b)\}$ is an open set. **(4 marks)**

(iii) (a) Let $A \subseteq \mathbb{R}^2$ be an open set and let (a,b) be surrounded by A. Let $F\colon A\backslash\{(a,b)\} \to \mathbb{R}$ be a function. Define what it means for F to have *limit* L at $(a,b) \in A$. **(3 marks)**

(b) Prove that for any $x, y \in \mathbb{R}$

$$2|xy| \leqslant x^2 + y^2.$$

(3 marks)

(c) Let $A = \mathbb{R}^2 \backslash \{(0,0)\}$ and define $F\colon A \to \mathbb{R}$ by

$$F(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \text{ for } (x,y) \neq (0,0).$$

Show that F has limit 0 at $(0,0)$. **(3 marks)**

3 (a) Define what it means for a function $F\colon A \to \mathbb{R}$ defined on an open set $A \subseteq \mathbb{R}^2$ to be *differentiable* at $(a,b) \in A$. **(3 marks)**

(b) Let $A = \{(x,y) \mid y > 0\}$. Define $F\colon A \to \mathbb{R}$ by $F(x,y) = \dfrac{x}{y}$. Show directly from your definition in (a) that F is differentiable at every $(a,b) \in A$. You may use the result of 2(iii)(b) if you wish. **(14 marks)**

(c) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0); \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Find $\dfrac{\partial F}{\partial x}(0,0)$ and $\dfrac{\partial F}{\partial y}(0,0)$ and show that F is not differentiable at $(0,0)$.

State clearly and in full, but do not prove, any result about limits of composites which you use. **(8 marks)**

4 (a) State carefully and in full the version of the Implicit Function Theorem which applies to maps with domain an open set $A \subseteq \mathbb{R}^3$ and target $\mathbb{R}^2$ and which gives conditions for a solution in terms of z. **(10 marks)**

Consider the simultaneous equations

$$x^2 + 2x + z^2 = 0, \qquad x^2 + 4x + y^2 + z^2 = 0. \tag{$*$}$$

The point $(0, 0, 0)$ is a solution.

(b) Define a map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ so that solutions of $(*)$ correspond to solving $\varphi(x, y, z) = (0, 0)$. **(1 mark)**

(c) Show that the hypotheses of the Implicit Function Theorem, as you stated it in (a), are not satisfied at $(0, 0, 0)$. **(2 marks)**

(d) Solve $(*)$ explicitly for x and y in terms of z, and identify those solutions which pass through $(0, 0, 0)$. Specify an open interval $I \subseteq \mathbb{R}$ containing 0, such that your solutions are valid for $z \in I$. **(5 marks)**

(e) Explain in one or two short sentences why your solution in (d) is not of the kind guaranteed by the Implicit Function Theorem. **(3 marks)**

(f) Verify that

$$x = -1 + \cos t, \quad y = 2 \sin \tfrac{1}{2} t, \quad z = \sin t, \tag{$\dagger$}$$

satisfies the equations $(*)$ for all $t \in \mathbb{R}$.

Explain in one or two short sentences why $(\dagger)$ is not a solution of the kind guaranteed by the Implicit Function Theorem. **(4 marks)**

5 (a) Let A and B be open subsets of $\mathbb{R}^2$. Define what it means for a map $\varphi\colon A \to B$ to be a diffeomorphism.

For φ a diffeomorphism, write down the matrix for $D(\varphi^{-1})(u,v)$ in terms of partial derivatives of φ. **(6 marks)**

(b) State carefully and in full the Local Diffeomorphism Theorem for maps $A \to B$ where A and B are open sets in $\mathbb{R}^2$. **(6 marks)**

(c) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function for which both $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$ are also C^1 functions. Define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi(x,y) = (F_x(x,y), F_y(x,y)).$$

State, in terms of partial derivatives of F, the condition which ensures that the Local Diffeomorphism Theorem can be applied to φ. **(3 marks)**

(d) Let F and φ be as in (c) and assume now that your condition, for the Local Diffeomorphism Theorem to be applied to φ, holds. Let A and B be open subsets of $\mathbb{R}^2$ such that $\varphi\colon A \to B$ is a diffeomorphism. Write $\varphi^{-1}(u,v) = (G(u,v), H(u,v))$ for the inverse map and define $W\colon B \to \mathbb{R}$ by

$$W(u,v) = -F(G(u,v), H(u,v)) + uG(u,v) + vH(u,v).$$

Find $\dfrac{\partial W}{\partial u}$ and $\dfrac{\partial W}{\partial v}$, simplifying your answers as much as possible.

(10 marks)

End of Question Paper