Data provided: Formulae sheet

CIV340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010-2011

Computational Engineering Mathematics — Three hours

Marks will be awarded for your best FOUR answers

1 (i) The second order PDE for function $U(x, y)$

$$A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU = 0,$$

where A, B, C, D, E and F are constants or functions of x, y, can be classified as being either elliptic, parabolic or hyperbolic according to the values of A, B and C. State the criterion for the classification and hence classify each of the following PDEs:

(a) $\frac{\partial^2 U}{\partial x^2} + 4\frac{\partial^2 U}{\partial x \partial y} + 4\frac{\partial^2 U}{\partial y^2} - \frac{\partial U}{\partial y} = f$;

(b) $2\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x \partial y} + 7\frac{\partial^2 U}{\partial y^2} = \frac{\partial U}{\partial x} + 5x^2$;

(c) $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y^2}$.

(d) $x\frac{\partial^2 U}{\partial x^2} - (x+1)\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2} = 0$.

(7 marks)

(ii) The one-dimensional diffusion equation is given by

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \quad (0 \le x \le 1),$$

where α is the diffusion coefficient. The equation is to be solved together with the necessary additional conditions:

$$U(x, 0) = f(x) \quad \text{and} \quad U(0, t) = a,\ U(1, t) = b.$$

Using the standard notation that $U_{ij} \equiv U(x_i, t_j)$ together with the conventions that $i = 0$ and $i = N$ correspond to $x = 0$ and $x = 1$, respectively, and that $j = 0$ corresponds to $t = 0$, use the standard finite difference approximations, given on the formulae sheet, together with the notation $k = \Delta t / \Delta x^2$, to derive the *explicit scheme*

$$U_{i,j+1} = \alpha k (U_{i+1,j} + U_{i-1,j}) + (1 - 2\alpha k) U_{ij}$$

for $i = 1, .., N-1$, $j = 0, 1, 2, ..$, which approximates the diffusion equation.

(3 marks)

(iii) Using the same notation and conventions, as well as the formulae sheet, derive the *implicit scheme* for the above diffusion equation

$$\alpha k U_{i+1,j} - (1 + 2\alpha k) U_{i,j} + \alpha k U_{i-1,j} = -U_{i,j-1},$$

for $i = 1, .., N-1$, $j = 1, 2, ...$

(4 marks)

 Question 1 continued on next page

1 (continued)

(iv) The diffusion equation is to be solved (approximately) over the range $0 \leq x \leq 1$ for the temperature distribution along a given steel billet with boundary conditions $U(0,t) = 0^oC$ and $U(1,t) = 50^oC$ and initial conditions $U(x,0) = 50x$, where it is assumed that the units have been normalized so that $\alpha = 1$. Assuming that we use $\Delta x = 0.25$ and $\Delta t = 0.1$, then use the *implicit* scheme to write down the system of algebraic equations for the temperature at $x = 0.25, 0.5, 0.75$ and time $t = 0.1$. Do NOT attempt to solve the system. ***(6 marks)***

2 (i) The elastic constitutive matrix for an isotropic material is given by

$$C = \begin{bmatrix} (\lambda+2\mu) & \lambda & \lambda & 0 & 0 & 0 \\ & (\lambda+2\mu) & \lambda & 0 & 0 & 0 \\ & & (\lambda+2\mu) & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & \text{sym} & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Given further that $E = 32.6GPa$, $\nu = 0.187$, and that a state of strain defined by $\varepsilon_{xx} = 1020 \times 10^{-6}$, $\varepsilon_{yy} = -0.28\varepsilon_{xx}$, $\varepsilon_{zz} = -0.19\varepsilon_{xx}$, $\varepsilon_{xy} = 237 \times 10^{-6}$, $\varepsilon_{yz} = 419 \times 10^{-6}$ and $\varepsilon_{zx} = -71 \times 10^{-6}$ exists at a point in a given isotropic material, calculate the corresponding state of stress at the point. ***(9 marks)***

(ii) Using the results in the first part, evaluate the following quantities:

- The mean stress $\sigma_{kk}/3$;
- The deviatoric stress defined by

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk},$$

  where δ_{ij} represents the (i,j) element of the usual 3×3 unit matrix (or called the Kronecker delta).
- $\sigma_{1k}\varepsilon_{k1}$ and $\sigma_{1k}\varepsilon_{k2}$.

Note that **a repeated index in any of** i**,** j **or** k **only** indicates that the summation convention is to be used, and that $\varepsilon_{11} = \varepsilon_{xx}$, $\varepsilon_{12} = \varepsilon_{xy}$, etc. ***(11 marks)***

3 (i) Consider a small volume of fluid in a cube with sides Δx, Δy and Δz.

(a) Draw a diagram showing only the projection of the volume on the $x - y$ plane, and sketch the x-components of the forces acting on the volume. ***(2 marks)***

(b) With the help of the diagram, derive the x-component of the equations of momentum conservation for a three-dimensional moving fluid. ***(10 marks)***

(ii) Derive the expressions for the net rates of work done to the fluid volume by the x-component of the pressure force as well as the x-components of the stresses. ***(8 marks)***

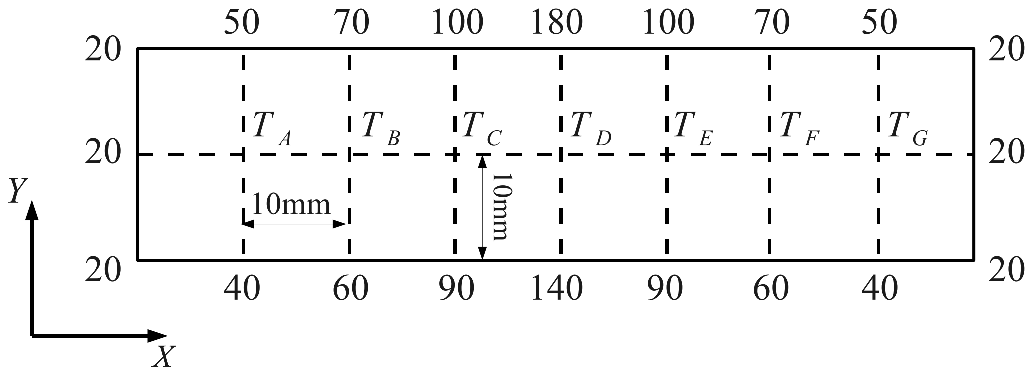

Figure 1: A rectangular plate with temperature defined on the boundaries.

4 Figure 1 shows a rectangular plate made of an homogeneous isotropic material. The temperature distribution in this plate satisfies the indicated boundary conditions (given in degrees centigrade) and has reached a steady-state condition so that the temperature is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

(i) Draw a sketch of the solution domain showing clearly the line of symmetry for the temperature distribution and indicating which of the unknown temperatures are equal to each other. ***(4 marks)***

(ii) Use the finite difference formulae on the formulae sheet to formulate the finite difference equations required to find estimates of the nodal temperatures, T_A, T_B, T_C and T_D. ***(9 marks)***

 Question 4 continued on next page

4 (continued)

(iii) Express these finite difference equations in the form $A\mathbf{T} = \mathbf{B}$ where A is a 4×4 matrix, $\mathbf{T} = (T_A,\ T_B,\ T_C,\ T_D)^T$ and $\mathbf{B} = (-110, -130, -190, -320)^T$ is a 4×1 column vector. Find matrix A, hence, given that

$$A^{-1} \approx \begin{bmatrix} -0.27 & -0.07 & -0.02 & -0.01 \\ -0.07 & -0.29 & -0.08 & -0.02 \\ -0.02 & -0.08 & -0.31 & -0.08 \\ -0.01 & -0.04 & -0.15 & -0.29 \end{bmatrix}$$

estimate T_A, T_B, T_C and T_D correct to one degree. ***(7 marks)***

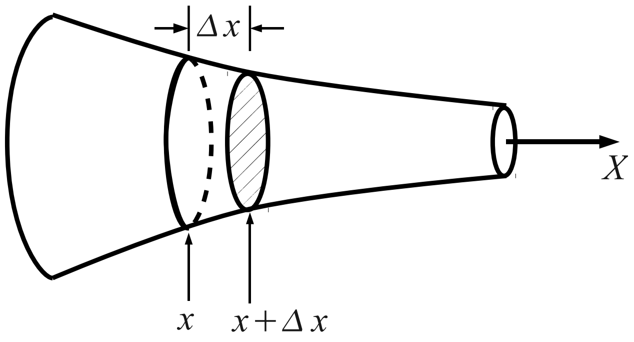

Figure 2: A one-dimensional bar with varying cross-section area.

5 (i) Consider a one-dimensional bar with a non-uniform cross-sectional area $A(x)$ (see Figure 2), which is loaded so that the normal stress in the x-direction at location x is $\sigma_x(x)$ and the body force per unit length in the x-direction is $f(x)$. By considering a layer with thickness Δx shown in the figure, show that the force balance equation for the stress $\sigma_x(x)$ and the body force $f(x)$ is

$$\frac{d[A(x)\sigma_x(x)]}{dx} + f(x) = 0\,.$$

Assuming simple stress-strain relation $\sigma_x(x) = E\epsilon_x(x)$, where $\epsilon_x(x) = du(x)/dx$ is the strain in x direction and E is the Young's modulus, write down the equation for the displacement $u(x)$. ***(5 marks)***

 Question 5 continued on next page

5 (continued)

(ii) Now assume $A(x)$, E, $f(x)$, and boundary conditions are given, so that the equation and boundary values for $u(x)$ can be simplified to

$$\mathcal{L}(u) \equiv (1+x)\frac{d^2u}{dx^2} + \frac{du}{dx} - 5 = 0, \quad \text{given } \; u(0) = 1 \text{ and } u(1) = 0.$$

The equation is to be solved by the weighted residual method.

(a) Given the two trial functions

$$\begin{aligned} U_1(x) &= x + c_1(x^2 - x) + c_2(x^3 - x), \\ U_2(x) &= (1-x) + c_1 x(1-x) + c_2 x(1-x^2), \end{aligned}$$

determine which one of them automatically satisfies the boundary conditions for $u(x)$. In the above equations, c_1 and c_2 are constants. ***(3 marks)***

(b) Determine the residual, $\mathcal{L}(U) = R(x)$, associated with your chosen trial function and then, by applying the weight functions $w(x) = 1$ and $w(x) = x$ in turn, use the condition

$$\int_0^1 w(x)\, R(x)\, dx = 0$$

to derive two algebraic equations for c_1 and c_2. Solve these equations working correct to four decimal places. ***(12 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$