The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010–2011

Continuum Mechanics — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The transformation matrix from the old to the new coordinates is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

Verify that $\hat{\mathbf{A}}\hat{\mathbf{A}}^T = \hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ is the unit matrix, so that $\hat{\mathbf{A}}$ is orthogonal.

(4 marks)

(ii) The matrix of components of the second-order tensor $\mathbf{T}$ is given in the old coordinate system by

$$\hat{\mathbf{T}} = \begin{pmatrix} 1 & a & 0 \\ 0 & b & c \\ 0 & 3 & 2 \end{pmatrix}$$

Calculate the matrix $\hat{\mathbf{T}}'$ of components of this tensor in the new coordinates using the relation $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$.

(8 marks)

(iii) You are given that the tensor $\mathbf{T}$ is invariant under the coordinate transformation determined by matrix $\hat{\mathbf{A}}$, i.e. the matrix of components of tensor $\mathbf{T}$ is the same in the old and new coordinate systems. Find a, b and c.

(13 marks)

2 (i) Give the definition of a particle trajectory. Show that, in Cartesian coordinates x_1, x_2, x_3, the system of equations for trajectories can be written in the form

$$\frac{dx_1}{dt} = v_1(\mathbf{x}, t), \quad \frac{dx_2}{dt} = v_2(\mathbf{x}, t), \quad \frac{dx_3}{dt} = v_3(\mathbf{x}, t),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity and $\mathbf{x} = (x_1, x_2, x_3)$. ***(6 marks)***

(ii) The velocity field of a planar motion is given by

$$\mathbf{v} = \left[\frac{ax_1(x_1^2 + x_2^2)^{1/2}}{T(a^2 + x_1^2 + x_2^2)} + \frac{tx_2(a^2 + x_1^2 + x_2^2)}{a^2(t^2 + T^2)}\right]\mathbf{e}_1 + \left[\frac{ax_2(x_1^2 + x_2^2)^{1/2}}{T(a^2 + x_1^2 + x_2^2)} - \frac{tx_1(a^2 + x_1^2 + x_2^2)}{a^2(t^2 + T^2)}\right]\mathbf{e}_2, \qquad (*)$$

where a and T are positive constants, and $\mathbf{e}_1$, $\mathbf{e}_2$ are the base vectors.

(a) Use the variable substitution

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi$$

to obtain the system of equations defining trajectories in the x_1x_2-plane in polar coordinates r, ϕ for the velocity field $(*)$. ***(9 marks)***

(b) The particle is at distance a from the coordinate origin when $t = 0$. Determine the particle distance from the origin at $t = T$. ***(10 marks)***

3 (i) Give the definitions of the principal directions and principal stresses of the stress tensor $\mathbf{T}$, and write down the equation defining the principal directions and stresses. ***(5 marks)***

(ii) At a point P the matrix of the components of the stress tensor $\mathbf{T}$ is given in Cartesian coordinates x_1, x_2, x_3 by

$$\hat{\mathbf{T}} = \begin{pmatrix} 5 & 0 & 6 \\ 0 & 5 & 8 \\ 6 & 8 & a \end{pmatrix} \text{Nm}^{-2}.$$

(a) You are given that one of the principal stresses is equal to 25. Use this condition to determine a. Calculate two other principal stresses. ***(9 marks)***

(b) Find the unit vectors in the principal directions of stress. ***(11 marks)***

4 (i) Using Euler's equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\mathbf{v}\|^2 + \varphi\right),$$

where φ is the body force potential, derive Bernoulli's integral for fluid stationary motion,

$$p + \frac{\rho}{2}\|\mathbf{v}\|^2 + \rho\varphi = \text{const.}$$

(8 marks)

(ii) Using Bernoulli's integral for an immovable fluid show that the water pressure is given by

$$p = p_a - \rho g z,$$

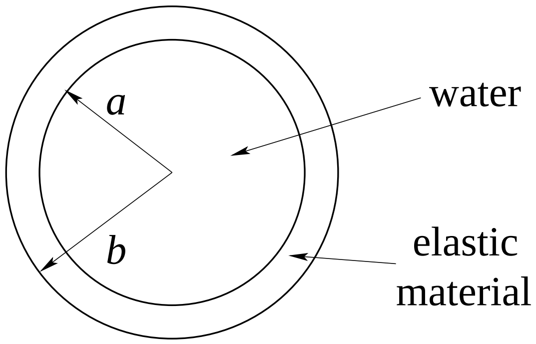
where p_a is the atmospheric pressure, ρ the water density, g the gravitational acceleration, and the z-axis is directed upwards with $z = 0$ at the water surface. ***(4 marks)***

(iii) Prove Archimedes' law: the pressure force exerted on the surface of a body immersed in water is in the vertical direction, and its magnitude is equal to the weight of water displaced by the body. ***(8 marks)***

(iv) A bathysphere has the shape of a sphere of radius $r = 1\,\text{m}$. It is completely immersed in water and attached to a ship by a steel rope. The tension in the rope is $T = 10^4$ N. What is the mass of the bathysphere? (You can take the water density $\rho = 10^3$ kg/m^3 and $g = 10$ m/s^2.) ***(5 marks)***

5 (i) Write down the expression for the surface traction, $\mathbf{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\mathbf{n}$. ***(2 marks)***

(ii) There is an elastic spherical shell of internal radius a and external radius b. The space inside the shell, which is the sphere of radius a (see the figure), was filled with water through a small hole, and after that the hole was tightly sealed. Then the shell was put in a cold place with the temperature below zero. As a result the water froze and turned into ice. Calculate the internal radius of the shell, R, after the water froze. (You can take the densities of water and ice equal to $\rho \approx 1000\ \mathrm{kg\ m^{-3}}$ and $\rho_i \approx 917\ \mathrm{kg\ m^{-3}}$ respectively.) ***(5 marks)***



(iii) You are given that the displacement in the shell satisfies

$$\nabla_\xi(\nabla_\xi \cdot \mathbf{u}) = 0.$$

Show that the displacement in the shell is given by

$$u = Ar + \frac{B}{r^2},$$

where A and B are constants. (You can assume that $\mathbf{u} = u(r)\mathbf{e}_r$ in the spherical coordinate system r, θ, ϕ with the origin at the shell centre, and use without proof that $\nabla_\xi \cdot (u(r)\mathbf{e}_r) = \dfrac{1}{r^2}\dfrac{d(r^2 u)}{dr}$.) ***(5 marks)***

(iv) You are given that the surface traction at the external boundary of the shell is given by

$$\mathbf{t}(b) = \left(\frac{\lambda}{r^2}\frac{d(r^2 u)}{dr}\bigg|_{r=b} + 2\mu \frac{du}{dr}\bigg|_{r=b} \right) \mathbf{e}_r.$$

Use the boundary conditions $u = R - a$ at $r = a$ and $\mathbf{t} = 0$ at $r = b$ to calculate A and B, where R was calculated in part (ii). Determine the external radius of the deformed shell if $a = 10$ cm, $b = 12$ cm, and $\lambda = \mu$. ***(13 marks)***

End of Question Paper