The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010–11

WAVES — 2 hours

*Marks will be awarded for your best* FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

**1** (i) *Derive* d'Alembert's general solution for the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

on $-\infty < x < \infty$ for $t \geq 0$.

***(13 marks)***

(ii) Given that $c = 1$ and at $t = 0$

$$\phi(x, 0) = \cos kx, \qquad \frac{\partial \phi}{\partial t} = k \sin kx,$$

where $k$ is a constant, find $\phi(x, t)$.

***(7 marks)***

(iii) Give a physical interpretation of your solution. Further, explain why the solution, subject to the initial conditions in (ii), cannot (or can) be a standing wave.

***(5 marks)***

**2** A uniform finite string of length $L$ and uniform density $\rho$ undergoes small transverse vibrations with displacement $y(x, t)$, where $y_{tt} = c^2 y_{xx}$, and $c^2$ is a constant.

(i) Given that $y(0, t) = y(L, t) = 0$, *derive* by using the method of separation of variables, or otherwise, that the general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right),$$

where $\{a_n\}$, $\{b_n\}$ are constants. ***(15 marks)***

 Question 2 continued on next page

2 (continued)

(ii) Find $\{a_n\}$ and $\{b_n\}$ for the case when

$$y(x,0) = A\sin\left(\frac{4\pi x}{L}\right); \qquad y_t(x,0) = 0,$$

where A is constant.

***(10 marks)***

3 In a compressible fluid the equilibrium density and pressure are $\rho_0$ and $p_0$ respectively, where $\rho_0$ and $p_0$ are uniform and constant. Due to the passage of a sound disturbance, there are *small* changes in density $\rho$ and pressure $p$, and the resulting fluid velocity is $\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}$.

(i) Given that the exact equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

obtain a valid approximation to this equation when $(\rho - \rho_0)$, $u$, $v$ and $w$ are small.

***(4 marks)***

(ii) Newton's Second Law can be approximated by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

plus two similar equations. Given that $p$ is a function only of $\rho$, show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right),$$

where $c^2$ is a constant which should be defined.

***(8 marks)***

(iii) In a particular case $\rho = \rho(r, t)$, where (as usual) $r^2 = x^2 + y^2 + z^2$. Show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial r^2} + \frac{2}{r}\frac{\partial \rho}{\partial r}\right) \qquad (*)$$

Find the partial differential equation satisfied by $R = r\rho$, and hence write down the general solution of $(*)$. (You may quote d'Alembert's general solution of the one-dimensional wave equation.)

***(13 marks)***

**4** The equilibrium position of the free surface of a liquid of infinite depth is $z = 0$, where $z$ is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where $x$ is measured along the undisturbed surface and

$$\eta = a\cos(kx - \omega t),$$

with $a$, $k$ and $\omega$ being positive constants with $a$ small. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\phi_{xx} + \phi_{zz} = 0.$$

You are also given that (a) $\phi_z \to 0$ as $z \to -\infty$; (b) $\phi_z = \eta_t$ at $z = 0$; (c) $\phi_t + g\eta = 0$ at $z = 0$.

(i) Give a brief physical interpretation of (a), (b) and (c).

***(6 marks)***

(ii) Find $\phi(x, z, t)$ and show that $\omega^2 = gk$.

***(13 marks)***

(iii) Determine the phase velocity $c$ and the group velocity $c_g$ in terms of $k$. Show that $c_g = c/2$, and state what is the speed of the propagation of energy.

***(6 marks)***

**5** In a model of traffic flow in the direction of $Ox$, the density of traffic at time $t$ is $\rho(x, t)$, and it is assumed that the velocity of traffic of density $\rho$ is $v = v(\rho)$.

(i) Show that

$$\rho_t + c(\rho)\rho_x = 0,$$

where $c(\rho) = d(\rho v)/d\rho$.

***(4 marks)***

(ii) Given that $\rho = f(x)$ at $t = 0$ for $-\infty < x < \infty$, and that

$$c(f(\xi)) = F(\xi),$$

show that, for $t \geq 0$, $\rho = f(\xi)$ on the curve $x = \xi + F(\xi)t$.

***(7 marks)***

 **Question 5 continued on next page**

**5** (continued)

(iii) Show that the above solution breaks down on any curve for which $F'(\xi) < 0$.

***(2 marks)***

(iv) In a particular case

$$v(\rho) = \frac{V}{P}(P - \rho) \qquad (0 \le \rho \le P);$$

where $V$ and $P$ are constant. Given that

$$\rho(x, 0) = \begin{cases} 0 & (x \le 0), \\ \rho_R(x^2/L^2) & (0 \le x \le L) \\ \rho_R & (x \ge L), \end{cases}$$

where $L$ and $\rho_R$ are constants with $\rho_R < P$, obtain the solution to the traffic flow equation in (i). Determine when and where the solution first breaks down.

***(12 marks)***

## End of Question Paper