The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-11**

Mathematical Methods **2 hours**

*Marks will be awarded for your best **FOUR** answers. The marks awarded to each question or section of question are shown in italics.*

1 The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx,$$

with corresponding inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k)\, dk.$$

The convolution, $F(x)$, of functions $f(x)$ and $g(x)$ is defined by

$$F(x) = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds.$$

(a) Show that

$$\hat{F}(k) = \hat{f}(k)\, \hat{g}(k).$$

(7 marks)

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} 1 & |x| \leqslant 1 \\ 0 & |x| > 1. \end{cases}$$

(i) Find $\hat{f}(k)$. ***(3 marks)***

(ii) Find the convolution of f with itself. ***(11 marks)***

Hence (using the result in part (a)) deduce that $4\sin^2 k/(k^2)$ has inverse Fourier transform given by:

$$\begin{cases} x+2 & -2 < x < 0 \\ 2-x & 0 < x < 2 \\ 0 & |x| > 2. \end{cases}$$

(4 marks)

2 The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} + 3\dot{x} + 2x = f(t),$$

for some function $f(t)$, with $x(0) = \dot{x}(0) = 1$.

(a) By taking the Laplace transform of the equation, show that

$$\tilde{x}(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\tilde{f}(s) + \frac{3}{s+1} - \frac{2}{s+2},$$

where $\tilde{x}(s)$ is defined by

$$\tilde{x}(s) = \int_0^\infty e^{-st}\, x(t)\, dt,$$

and $\tilde{f}(s)$ is defined similarly. ***(9 marks)***

Hence derive the solution

$$x(t) = 3e^{-t} - 2e^{-2t} + \int_0^t f(u)\left\{e^{-(t-u)} - e^{-2(t-u)}\right\} du.$$ ***(5 marks)***

> You may assume that the following hold:
>
> $$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n\tilde{x}(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \cdots - x^{(n-1)}(0)$$
>
> $$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \quad \text{for} \quad \operatorname{Re} s > a,$$
>
> $$\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \tilde{f}(s)\tilde{g}(s).$$
>
> where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.

(b) Use the result of part (a) to find the solution $x(t)$ when $f(t) = e^{-t}$. ***(5 marks)***

Verify that this solution does satisfy the differential equation and the initial conditions. ***(6 marks)***

3 The function $y(x)$ satisfies the ordinary differential equation

$$x^2y'' - 2y = 3x^3e^{-x}$$

in $0 < x < \infty$, with the boundary conditions that y is finite at $x = 0$ and as $x \to \infty$.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2y'' - 2y = 0.$$ ***(3 marks)***

(b) Given that Green's function, $G(x;\xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} -\dfrac{x^2}{3\xi^3} & 0 \leqslant x < \xi, \\ \\ -\dfrac{1}{3x} & \xi < x < \infty. \end{cases}$$ ***(10 marks)***

(c) Using Green's function, show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = 3(x+2)e^{-x} - \frac{6}{x}\left(1 - e^{-x}\right).$$ ***(12 marks)***

4 Consider the equation

$$\epsilon x^3 - x^2 + 1 = 0, \qquad (1)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (1) can be written as

$$x = \frac{1}{\epsilon}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Substitute this expression into equation (1) and hence derive the solutions

$$x \sim \begin{cases} \pm 1 + \dfrac{1}{2}\epsilon + O(\epsilon^2) \\ \dfrac{1}{\epsilon} - \epsilon + O(\epsilon^2) \end{cases}$$

as $\epsilon \to 0$. ***(18 marks)***

(b) Given the rearrangement

$$x = \left(1 + \epsilon x^3\right)^{1/2}$$

of equation (1), use iteration to find the solution close to 1, correct to $O(\epsilon^2)$ as $\epsilon \to 0$. ***(7 marks)***

5 The *exponential integral* is defined by

$$E(x) = \int_1^\infty t^{-1} e^{-xt}\, dt \qquad \text{for} \qquad x > 0.$$

(a) Show that

$$e^x E(x) = \int_0^\infty \frac{e^{-xv}}{1+v}\, dv.$$ ***(3 marks)***

Use the sum of a geometric progression to show that

$$\frac{1}{1+v} = 1 - v + v^2 - v^3 + \cdots + (-v)^{n-1} + \frac{(-v)^n}{1+v}.$$ ***(3 marks)***

By using the above results and considering

$$I_n(x) = \int_0^\infty v^n\, e^{-xv}\, dv \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

deduce that

$$e^x E(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + R_n(x),$$

where

$$R_n(x) = (-1)^n \int_0^\infty \frac{v^n\, e^{-xv}}{1+v}\, dv.$$ ***(11 marks)***

(b) By considering

$$\left| \frac{R_n(x)}{\frac{(-1)^{n-1}(n-1)!}{x^n}} \right|$$

as $x \to \infty$, show that $E(x)$ has the asymptotic series

$$E(x) \sim e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^n\, n!}{x^{n+1}} + \cdots \right)$$

as $x \to \infty$. ***(8 marks)***

End of Question Paper