The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

MAS331 Metric Spaces **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Give a precise definition of what it means for d to be a *metric* on a set X. ***(4 marks)***

(ii) The Euclidean metric d_2 on $\mathbb{R}^n$ is defined by the prescription

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2},$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$.

Prove that d_2 really is a metric. ***(10 marks)***

[Hint. You may use the Cauchy-Schwarz inequality:

$$\left|\sum_{i=1}^n a_i b_i\right| \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2},$$

for $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$.]

(iii) Consider $\mathbb{R}^3$ with the metric d_2. Show that the sequence whose nth term is (x_n, y_n, z_n) converges to (x, y, z) as $n \to \infty$ in the metric space $(\mathbb{R}^3, d_2)$ if and only if $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$. ***(5 marks)***

(iv) Find the limit in $(\mathbb{R}^3, d_2)$ of the sequence whose nth term is

$$\left(\cos\left(\frac{2}{n+1}\right), \ln\left(\frac{n+5}{n}\right), \frac{3n+1}{n+2}\right),$$

and calculate the distance between this limit and the origin. ***(6 marks)***

2 (i) Define the *supremum* or d_∞ metric on the space $C[0,1]$ of all continuous functions from $[0,1]$ to $\mathbb{R}$. ***(1 mark)***

(ii) Explain what it means for a sequence (f_n) of functions in $C[0,1]$ to converge *pointwise* to $f \in C[0,1]$. ***(2 marks)***

(iii) Prove that if the sequence (f_n) converges to f in the metric space $(C[0,1], d_\infty)$ then it also converges to f pointwise. ***(5 marks)***

(iv) For each of the following sequences of functions, deduce whether convergence takes place pointwise, in $(C[0,1], d_\infty)$, both or neither?

(a) $f_n(x) = \dfrac{1}{2} + \dfrac{1}{2}(1-x)^n.$ ***(5 marks)***

(b) $f_n(x) = \cos\left(\dfrac{(x+2n)^2}{n^3}\right)$ ***(5 marks)***

[Hint: You may use the inequality $|1-\cos(y)| \leq |y|$ for $y \in \mathbb{R}$.]

(v) Let (x_n) be a sequence in a metric space (X,d). If (x_n) converges to x show that every subsequence (x_{n_k}) also converges to x. ***(4 marks)***

(vi) Give an example of a sequence in a metric space that contains two subsequences converging to different limits. ***(3 marks)***

3 (i) Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a function.

(a) Explain in terms of convergent sequences what it means for f to be *continuous.* ***(2 marks)***

(b) Let $A \subseteq Y$.

(α) What is meant by $f^{-1}(A)$? ***(1 mark)***

(β) What does it mean (in terms of convergent sequences) for A to be *closed*? ***(2 marks)***

(γ) Prove that if f is continuous and A is closed then $f^{-1}(A)$ is also closed. ***(5 marks)***

(ii) Define the mapping I from $(C[0,1), d_\infty)$ to $\mathbb{R}$ with its usual metric by

$$I(f) = \int_0^1 f(x)dx.$$

(a) Show that I is continuous. ***(5 marks)***

(b) Deduce that the set $I^{-1}([0,1])$ is closed. ***(2 marks)***

(c) Which of the following functions are in the set $I^{-1}([0,1])$? Justify your answers.

(α) $f(x) = e^{-x}$. ***(4 marks)***

(β) $f(x) = (x+1)^2$. ***(4 marks)***

4 (i) (a) Give a precise mathematical definition of what it means for a sequence in a metric space (X, d) to be a *Cauchy sequence.*

(2 marks)

(b) Explain what it means for (X, d) to be *complete.* ***(1 mark)***

(ii) Let (f_n) be the sequence in $(C[0,1], d_\infty)$ defined for each $x \in [0,1]$ by

$$f_n(x) = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{3^n}.$$

(a) Deduce that for $m > n$,

$$d_\infty(f_n, f_m) = \frac{1}{2}\left(\frac{1}{3^n} - \frac{1}{3^m}\right)$$

(7 marks)

(b) Show that (f_n) is a Cauchy sequence. ***(3 marks)***

(c) Does (f_n) converge? If so, why? ***(2 marks)***

(iii) Explain what it means for a subset of a metric space to be

(a) *compact,* ***(1 mark)***

(b) possess the *Heine-Borel property.* ***(1 mark)***

Specify any relationship that holds between these two concepts

(2 marks)

(iv) Prove that the union of a finite number of compact sets is compact. Does this result also hold for infinite unions? Give a justification if your answer is yes and a counter-example if your answer is no. ***(6 marks)***

5 (i) Let (X, d) be a metric space. Explain what it means for a mapping $f : X \to X$

(a) to be a *contraction*, ***(2 marks)***

(b) to have a *fixed point*. ***(1 mark)***

(ii) Let $f : X \to X$ be a contraction of a complete metric space with metric d. Fix $x_0 \in X$ and define a sequence (x_n) by $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$

(a) Deduce that there exists $0 \leq k < 1$ such that

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0).$$

(4 marks)

(b) Prove that for $m > n$,

$$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_1, x_0),$$

and hence show that (x_n) is a Cauchy sequence. ***(6 marks)***

(c) Explain why the sequence (x_n) has a limit x and show that x is the *unique* fixed point of f. ***(7 marks)***

(iii) Consider the set $\mathbb{R}^2$ equipped with the Euclidean metric d_2. Show that f is a contraction where for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \left(\frac{1}{6} \cos(3y) + 1, \frac{1}{3} \cos(2x) - 3 \right).$$

(5 marks)

[Hint: You may use the fact that

$$|\cos(a) - \cos(b)| \leq |a - b|$$

for all $a, b \in \mathbb{R}$.]

End of Question Paper