The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010-2011**

Complex Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Express both of the following in the form $x + iy$:

$$\frac{25+5i}{2-i}\ ; \qquad (1-i)^{17}\ .$$ ***(4 marks)***

(ii) Express

$$\frac{(\sqrt{3}+i)^{17}}{(1-i)^{29}}$$

in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$. ***(4 marks)***

(iii) State, without proof, the triangle inequalities for $|\,z+w|$ and $|\,z-w|$. Show that, if $|z+4i| \ \leq \ 2$, then

$$3 \ \leq \ |\,z-3| \ \leq \ 7$$

and

$$\frac{1}{2} \ \leq \ \left|\frac{z-3}{z+8i}\right| \ \leq \ \frac{7}{2}\ .$$

(6 marks)

(iv) Find all the solutions of each of the following equations:

(a) $e^z + 1 \ = \ 0$,
(b) $2\cos z + 3i\sin z \ = \ 2$. ***(5 marks)***

(v) The path γ is the semicircle given by $z \ = \ e^{it} \quad (0 \ \leq \ t \ \leq \ \pi)$. Evaluate

$$\int_\gamma \operatorname{Im} z\, dz\,, \qquad \int_\gamma \frac{z}{(4+z^2)^2}\, dz\,.$$ ***(6 marks)***

2 (i) Define what is meant by the following three statements:

(a) D is a **region** in the complex plane;

(b) A function f is **differentiable at the point** z_0 ;

(c) A function f is **analytic in a region** D. ***(4 marks)***

Let

$$g(z) = \frac{e^z}{(1 - e^{\pi i z})} \, .$$

Decide where g is analytic giving reasons for your answer. ***(3 marks)***

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(2 marks)***

In each of the following cases, determine whether there is a function h analytic on $\mathbb{C}$ with $\operatorname{Re}(h(x+iy)) = u(x,y)$, giving reasons for your answers:

$$\begin{aligned} &\text{(a)} \quad u(x,y) = x^2 - 4xy - y^2 + 1 \, , \\ &\text{(b)} \quad u(x,y) = e^x \cosh y \, . \end{aligned}$$

When h exists, find an explicit expression for $h(z)$ in terms of z and show that you have all functions satisfying the conditions. ***(8 marks)***

(iii) Let the path α from 2 to -2, consist of the straight line segment from 2 to $2+2i$, followed by the straight line segment from $2+2i$ to $-2+2i$, followed by the straight line segment from $-2+2i$ to -2. Sketch α. Use the ML estimate to show that

$$\left| \int_\alpha \frac{\cos z}{z^2} \, dz \right| \leq 2 \cosh 2 \, .$$

(8 marks)

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let γ be the circular contour $|z+2| = 3$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{z^3\, e^z}{2z-1}\, dz$, (ii) $\displaystyle\int_\gamma \frac{\cos z}{(e^z+1)^2}\, dz$,

(iii) $\displaystyle\int_\gamma \frac{\sin \pi z}{(2z-1)^5}\, dz$, (iv) $\displaystyle\int_\gamma \mathrm{Im} z\, dz$,

(v) $\displaystyle\int_\gamma \frac{(z+1)\, e^z}{z^2-4}\, dz$.

(18 marks)

4 (i) Show that $\dfrac{1}{(z-1)(z+2)}$ has a Taylor series expansion valid in the disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Find the Taylor series giving an expression for the general term. ***(6 marks)***

(ii) For each of the following functions, find **all the singularities in** $\mathbb{C}$, classify these singularities giving reasons for you answers and evaluate the residue at each of them:

(a) $\dfrac{1-\cos z}{z^5}$, (b) $\dfrac{e^{2z}}{1-e^z}$, (c) $\dfrac{1}{z^3(z+1)}$. ***(11 marks)***

(iii) Find **all the singularities in** $\mathbb{C}$ of each of the following functions:

(d) $z \cos\left(\dfrac{1}{z-2}\right)$, (e) $\dfrac{z^2 \sin(\pi z)}{(e^{\pi i z}-1)^3}$.

and classify these singularities giving reasons for your answers. ***(8 marks)***

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

The contour γ consists of the real axis from the origin to the point 2, followed by the quarter circle from 2 to the point $2i$ given by $z = 2e^{it}$ $(0 \le t \le \frac{\pi}{2})$, followed by the straight line segment from $2i$ to the origin. Sketch γ and evaluate

$$\int_\gamma \frac{dz}{z^4+1} \, .$$ ***(7 marks)***

(ii) Let $\alpha > 0$. Prove that,

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x) + \sin(\alpha x)}{x^2+2x+5}\, dx = \frac{\pi}{2\, e^{2\alpha}} \, (\cos\alpha - \sin\alpha)\, .$$ ***(12 marks)***

What is the value of

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x) + \sin(\alpha x)}{x^2+2x+5}\, dx$$

when $\alpha < 0$? **Give reasons for your answer.** ***(2 marks)***

End of Question Paper