The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–2011**

Differential Geometry **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I and J denote open intervals in $\mathbb{R}$. You may assume all curves, surfaces and functions are smooth.

A sheet of Basic Formulas for use as needed is given on the final page.

1 (i) The curve

$$\beta(t) = (\cos t, \ln(\sec t + \tan t) - \sin t), \quad t \in (0, \tfrac{\pi}{2})$$

is called the *tractrix*.

(a) Calculate $|\beta'(t)|$ and deduce that β is a regular curve. **(5 marks)**

(b) Show that for all $t \in (0, \frac{\pi}{2})$, the distance from $\beta(t)$ to the point of intersection of the tangent line to β at t with the y–axis, is 1. **(5 marks)**

(c) Determine a unit–speed reparametrization of β, simplifying your answer as much as possible, and being sure to specify the domain. **(9 marks)**

(ii) Let $\alpha\colon I \to \mathbb{R}^2$ be a unit–speed curve. Using the fact that $\{\mathbf{T}(s), \mathbf{P}(s)\}$ is an orthonormal basis for $\mathbb{R}^2$, for all $s \in I$, or otherwise, prove that

$$\mathbf{P}'(s) = -\kappa(s)\mathbf{T}(s).$$

(6 marks)

2 (i) Verify that the following curves are regular and calculate their curvature, simplifying your answers as far as possible.

$$\beta(t) = (e^{-t}\cos t, e^{-t}\sin t), \qquad t > 0;$$

$$\gamma(t) = (\cos 2t + \cos t + 1, \sin 2t + \sin t), \qquad t \in \mathbb{R}.$$

(12 marks)

(ii) (a) Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a smooth function. Choose $s_0 \in I$ and define

$$\theta(s) = \int_{s_0}^{s} f(u)\, du.$$

Show that

$$\alpha(s) = \left(\int_{s_0}^{s} \cos\theta(u)\, du, \int_{s_0}^{s} \sin\theta(u)\, du\right)$$

is a smooth unit–speed curve $I \to \mathbb{R}^2$ with curvature $\kappa = f$.

(5 marks)

(b) Find a unit–speed curve $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ with curvature given by

$$\kappa(s) = \frac{1}{1+s^2}.$$

Give the curve explicitly, evaluating all integrals and reducing to simplest terms. **(8 marks)**

3 (i) Let $\rho\colon\ A \to \mathbb{R}^3$ be a surface S.

Show that $L = \mathbf{n}\,.\,\rho_{uu}$ and that $M = \mathbf{n}\,.\,\rho_{uv} = -\mathbf{n}_u\,.\,\rho_v$. **(5 marks)**

(ii) *Enneper's surface* is

$$\rho(u, v) = (u - \tfrac{1}{3}u^3 + uv^2,\ v - \tfrac{1}{3}v^3 + vu^2,\ u^2 - v^2), \qquad u, v \in \mathbb{R}.$$

You are given that the unit normal vector $\mathbf{n}$ for Enneper's surface is

$$\mathbf{n}(u, v) = \frac{1}{1 + u^2 + v^2}(-2u,\ 2v,\ 1 - u^2 - v^2).$$

Find the Gaussian curvature K and the mean curvature H of Enneper's surface.

(10 marks)

(iii) Let $\rho\colon\ A \to \mathbb{R}^3$ be a surface S.

(a) Define a, b by

$$-\mathbf{n}_u = a\rho_u + b\rho_v.$$

Prove that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L \\ M \end{bmatrix}.$$

(3 marks)

(b) Show that

$$|\rho_u \times \rho_v| = \sqrt{EG - F^2},$$

(4 marks)

(c) Using the fact that for any four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$,

$$(\mathbf{a} \times \mathbf{b})\,.\,(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\,.\,\mathbf{c})(\mathbf{b}\,.\,\mathbf{d}) - (\mathbf{a}\,.\,\mathbf{d})(\mathbf{b}\,.\,\mathbf{c}),$$

or otherwise, show that

$$K = \frac{\mathbf{n}\,.\,(\mathbf{n}_u \times \mathbf{n}_v)}{\sqrt{EG - F^2}}.$$

(3 marks)

4 Let $\rho\colon\ A \to \mathbb{R}^3$ be a surface S in $\mathbb{R}^3$.

(a) Suppose that every point of S is a scalar point, and that the principal curvature κ is constant at 0 over all of S. Prove that S is (part of) a plane. **(5 marks)**

(b) Suppose that every point of S is a scalar point, and that the principal curvature κ is constant at $a \neq 0$ over all of S. Prove that S is (part of) a sphere. **(12 marks)**

(c) Suppose merely that every point of S is a scalar point. Show that the principal curvature κ must be constant over all of S. Deduce that S is either (part of) a plane or (part of) a sphere. **(8 marks)**

5 A surface S called the *minimal monkey saddle* is given by

$$\rho(u,v) = \left(u - \tfrac{1}{5}(u^5 - 10u^3v^2 + 5uv^4),\ v + \tfrac{1}{5}(5u^4v - 10u^2v^3 + v^5),\ \tfrac{2}{3}(u^3 - 3uv^2)\right),$$

for $u, v \in \mathbb{R}$.

(a) Show that the unit normal to S is

$$\mathbf{n}(u,v) = \frac{1}{1+R^4}(2(v^2-u^2),\, 4uv,\, 1-R^8)$$

where $R^2 = u^2 + v^2$. **(10 marks)**

(b) Show that $L = 4u$, and also find N. **(10 marks)**

(c) You are given that

$$E = G = (1+R^4)^2, \qquad F = 0, \qquad M = -4v.$$

Find the Weingarten matrix, and the Gaussian and mean curvatures. **(5 marks)**

End of Question Paper

Basic Formulas

The following basic formulas for curves and surfaces are given for use as needed:

Let $\alpha\colon\ I \to \mathbb{R}^2$ be a smooth regular plane curve. Then the curvature κ is given by

$$\kappa(t) = \frac{\alpha_1'(t)\alpha_2''(t) - \alpha_2'(t)\alpha_1''(t)}{((\alpha_1'(t))^2 + (\alpha_2'(t))^2)^{3/2}}.$$

The derivatives of the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{P}$ are given by

$$\mathbf{T}' = s'\kappa\mathbf{P}, \qquad \mathbf{P}' = -s'\kappa\mathbf{T},$$

where $s\colon\ I \to \mathbb{R}$ is an arc–length function.

Let $\rho\colon\ A \to \mathbb{R}^3$ be a smooth regular surface S, where A is an open connected subset of $\mathbb{R}^2$. The *unit normal vector* $\mathbf{n}$ to S is given by

$$\mathbf{n} = \frac{\rho_u \times \rho_v}{|\rho_u \times \rho_v|},$$

where $\rho_u = \dfrac{\partial\rho}{\partial u}$, $\rho_v = \dfrac{\partial\rho}{\partial v}$. The coefficients of the first and second fundamental forms are given by

$$E = |\rho_u|^2,\ F = \rho_u \cdot \rho_v,\ G = |\rho_v|^2; \quad L = -\mathbf{n}_u \cdot \rho_u,\ M = -\mathbf{n}_v \cdot \rho_u,\ N = -\mathbf{n}_v \cdot \rho_v.$$

The Weingarten matrix is $\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$. The principal curvatures of S are the eigenvalues of W. The Gaussian curvature K of S and the mean curvature H, are given by

$$K = \det W = \frac{LN - M^2}{EG - F^2}, \qquad H = \tfrac{1}{2}\operatorname{tr} W = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

The following trigonometric formulas may be used as needed:

$$\cos(x+y) = \cos x\cos y - \sin x \sin y, \qquad \sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cosh^2 x - \sinh^2 x = 1, \quad \cosh 2x = \cosh^2 x + \sinh^2 x, \quad \sinh 2x = 2\sinh x\cosh x,$$
$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$