The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-2011**

Mathematics (Computational Methods) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1.

Let $$A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -4 & 5 \end{pmatrix}.$$

(i) Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. (***6 marks***)

(ii) Verify that L^{-1} and U^{-1} have, respectively, the forms:

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & 1 & 0 \\ \frac{8}{21} & a & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{21} & \frac{4}{65} \\ 0 & \frac{5}{21} & b \\ 0 & 0 & \frac{21}{65} \end{pmatrix}$$

and find the values of a and b. (***4 marks***)

(iii) Explain how you would use the result of part (ii) to find A^{-1}. Given that it has the form:

$$A^{-1} = \frac{1}{65}\begin{pmatrix} 17 & 10 & 4 \\ 10 & c & 10 \\ 8 & 20 & 21 \end{pmatrix}$$

find the value of c. (***2 marks***)

(iv) Derive the formulae for the implicit numerical solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$. (***3 marks***)

(v) If we are given that, in the heat equation of part (iv), $c = 2$ and the boundary and initial conditions are $u(0,t) = 18$, $\frac{\partial u}{\partial t}(3,t) = 0$ for $t \geq 0$, and $u(x,0) = 18 - 12x + 2x^2$ for $0 \leq x \leq 3$, then letting the x-increment be $h = 1$ and the t-increment be $k = 0.5$, set up a table showing the values of u at the grid points for $t = 0$, and hence calculate the values of u at the grid points for $t = 0.5$. (***10 marks***)

2.

(i) Let $A = \begin{pmatrix} 10 & -2 & 0 & 0 \\ -1 & 10 & -1 & 0 \\ 0 & -1 & 10 & -1 \\ 0 & 0 & -2 & 10 \end{pmatrix}$, $B = \begin{pmatrix} 970 & 196 & 20 & 2 \\ 98 & 980 & 100 & 10 \\ 10 & 100 & 980 & 98 \\ 2 & 20 & 196 & 970 \end{pmatrix}$

Evaluate AB and hence or otherwise find A^{-1}. **(*5 marks*)**

(ii) State whether the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} = 4$$

is elliptic, hyperbolic or parabolic, justifying your answer with an appropriate calculation. **(*2 marks*)**

(iii) The solution of the equation in part (ii) is to be approximated in the rectangular region

$$\left\{ (x, y) : 0 \le x \le \frac{3}{2}, 0 \le y \le 1 \right\},$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = -1 \text{ when } x = 0, \ \frac{\partial u}{\partial x} = 2 \text{ when } x = \frac{3}{2},$$
$$u = x^2 - x + y \text{ when } y = 0 \text{ or } 1.$$

Taking $h = k = \frac{1}{2}$, draw a suitable grid for a numerical analysis of the problem, and mark on it the known values of u. Also indicate on your diagram an appropriate notation for the unknown values and any fictitious values you will require to use. **(*6 marks*)**

(iv) Write down equations relating the variables specified in part (iii) and, by eliminating any fictitious values, show that they are of the form $A\mathbf{u} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence, solve the equations for the unknown values of $\mathbf{u}$. **(*12 marks*)**

3.

(i) Give a brief description of a cubic spline, including a clear statement of the conditions that must be satisfied by the two formulae on either side of a datum point. (***6 marks***)

(ii) By deriving appropriate formulae from the requirements detailed in part (i), find the cubic spline which fits the following set of data:

x	0	1	2	3
$f(x)$	10	14	26	36

subject to the additional requirements that the **first** derivative at $x = 0$ is zero and the **second** derivative at $x = 3$ is zero.

(***19 marks***)

You may use without proof that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \quad \text{is} \quad \frac{1}{26}\begin{pmatrix} 15 & -4 & 1 \\ -4 & 8 & -2 \\ 1 & -2 & 7 \end{pmatrix}.$$

4.

Let $f(x, y) = x^2 + 13y^2 - 7xy - 6x + 18y + 40$.

(i) Starting from the initial point $(x, y) = (-9, -3)$, use one iteration of the method of steepest descent to find a point where the value of $f(x, y)$ is less than it is at the origin. Evaluate the function at this point. (***10 marks***)

(ii) Find the gradient vector for the second iteration of this method, and verify that it is at right angles to the previous descent direction. (***3 marks***)

(iii) Use one iteration of Newton's method, starting from $(x, y) = (-9, -3)$ to search for the minimum point of $f(x, y)$. Show that the point obtained is in fact the minimum point of the function and find its value there. (***12 marks***)

5.

(i) Give a brief description of integer programming, indicating its relationship to (well-known) linear programming. No calculations are required, but you should suggest circumstances where it would be appropriate to use this technique and you should explain one way of reducing the problem to a sequence of linear programming problems.
(***4 marks***)

(ii) A ship builder builds aircraft carriers, battleships, cruisers and destroyers. An aircraft carrier costs £a_j in year j, a battleship £b_j, a cruiser £c_j and a destroyer £d_j. Over a four year period the ship builder must build 4 aircraft carriers, 3 battleships, 7 cruisers and 5 destroyers. They can build up to 6 ships in any one year. You may assume that any ship is built during one calendar year. Set up an integer programming problem to help decide what to build and when in order to minimise the costs, but do not attempt to solve it. (***4 marks***)

For each of the additional (independent) requirements below set up appropriate constraints for your integer programming problem:

(a) There must be at least one cruiser and one destroyer built each year. (***2 marks***)

(b) Aircraft carriers and battleships cannot be built in the same year. (***4 marks***)

(c) At the end of each year the company must have built at least twice as many cruisers as battleships up to that point. (***3 marks***)

(d) If all 3 battleships are built in one year, then no destroyers can be built that year. (***4 marks***)

(e) If 2 or more aircraft carriers are built in any one year, then none may be built in the following year. (***4 marks***)

6.

A lorry is to be loaded with a selection of goods of 3 types. The total weight of the load must not exceed 30 tonnes. The goods have weights and values as shown in the following table:

Type	weight (tonnes)	value (£ hundreds)
1	9	32
2	7	24
3	5	17

Using the dynamic programming algorithm construct an appropriate table to find the combination of goods which gives the highest value for the load on the lorry.
(***25 marks***)

End of Question Paper