The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2010-2011

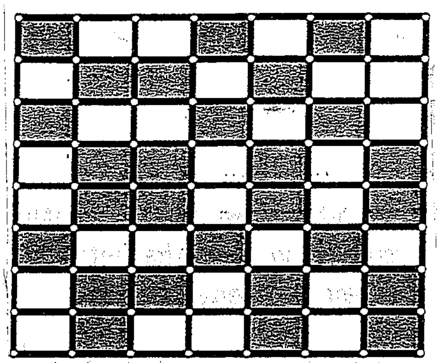
Graph Theory

2 hours and 30 minutes

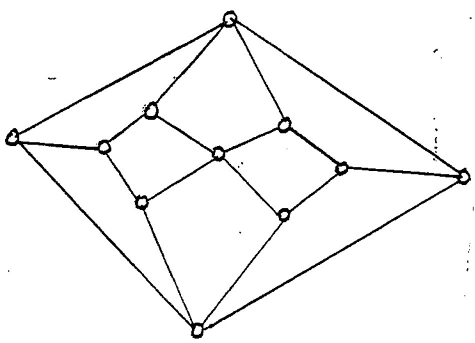
*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

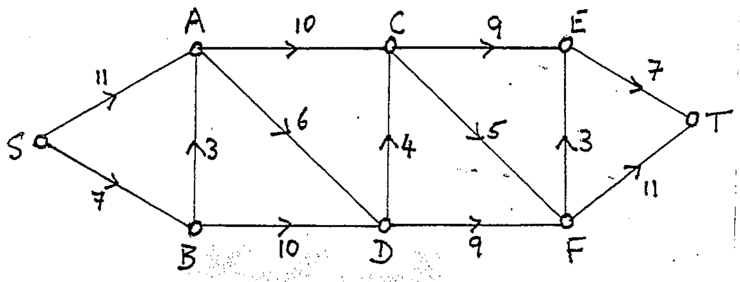
1 (i) Determine the number of non-isomorphic trees with exactly eight vertices, none of which has degree 2. *(5 marks)*

(ii) By drawing a suitable graph, determine whether the braced framework shown is rigid. If it is rigid, is it a minimum bracing? If it is not rigid, how many braces need to be inserted to make it rigid, and how many should *then* be removed to make it a minimum bracing? Justify all your answers graphically. *(9 marks)*



(iii) Is the graph shown below (a) Eulerian, (b) semi-Eulerian, (c) Hamiltonian, (d) semi-Hamiltonian? Justify your answers. *(11 marks)*





2 (i) Use the shortest and longest path algorithms to determine all shortest and longest paths from S to T in the network above, and state the lengths s and l of the shortest and longest paths. *(14 marks)*

The weight of each arc in turn is reduced by one unit, keeping the weights of all other arcs the same. For which arcs will this cause (a) s, (b) l to be lowered? If, instead, the weight of each arc in turn is raised by one unit, for which arcs will this cause (c) s, (d) l to be raised? *(6 marks)*

(ii) The distances between five towns $V - Z$ are given in the table shown.

V				
6	W			
5	4	X		
3	7	7	Y	
6	6	8	4	Z

By initially omitting V, find a good lower bound for the solution of the Travelling Salesman Problem for these towns, explaining how you obtained this. Why is this a solution of the Travelling Salesman Problem? *(5 marks)*

3 (i) State *Euler's Formula* for a plane, connected graph. *(1 mark)*

The plane, connected graph G has vertices of degrees 3 and 4 only, and every face has degree 4. Determine the number of vertices of degree 3. Give two examples of such a graph. *(12 marks)*

(ii) Explain what is meant by the *genus* of a graph, and state *Euler's Formula* for a connected graph of genus g. *(3 marks)*

Let g be the genus of a connected graph with $v \geq 3$ vertices and e edges, and with every circuit of length at least 4. Prove that

$$g \geq \frac{1}{4}e - \frac{1}{2}v + 1.$$

(4 marks)

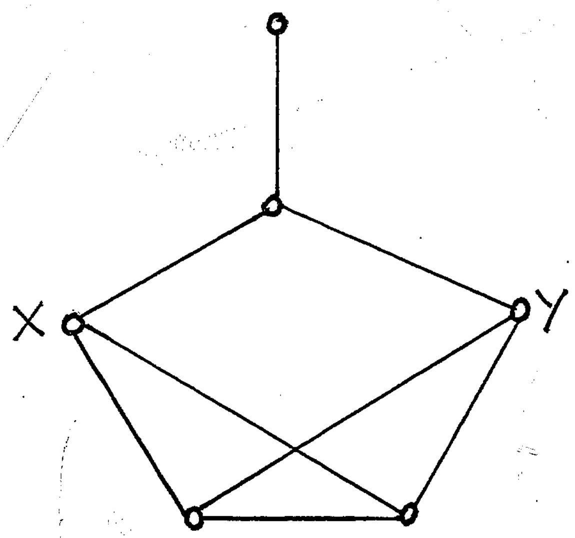
Deduce that the genus of the complete bipartite graph $K_{m,n}$ is at least $\frac{1}{4}(m-2)(n-2)$, conjecture what *is* the genus of $K_{m,n}$, and verify your conjecture for $K_{3,3}$. *(5 marks)*

4 (i) Eight students are to attend job presentations numbered 1 to 8 as given.

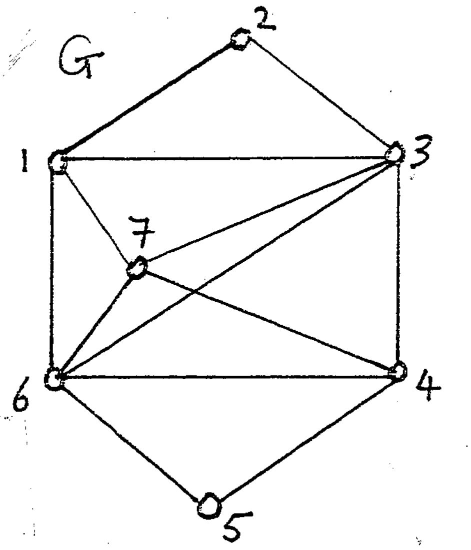
Anna:	1, 3, 4
Ben:	1, 2, 6
Cath:	1, 4, 5
Doug:	1, 2, 3
Emma:	4, 5, 7
Fay:	5, 6, 7
George:	2, 3, 8
Harry:	3, 4, 8

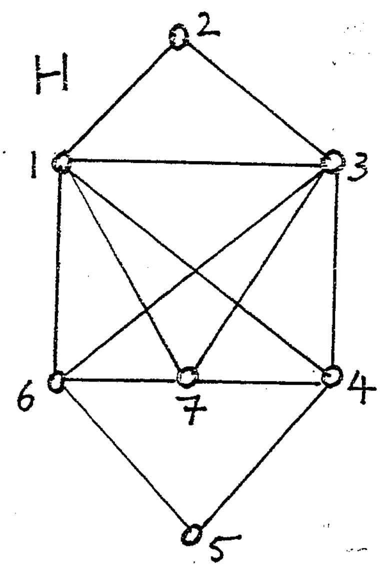
Each student attends only one presentation per day and each presentation is only given once. By drawing a suitable graph, use a property of this graph to obtain the smallest number of days in which the presentations can be given and describe one such way *(9 marks)*

(ii) Determine the chromatic polynomial of the graph shown below, and explain how to deduce the numbers of vertices and edges of the graph from this polynomial. *(10 marks)*



Determine the number of ways of vertex-colouring the graph using four given colours when (a) X, Y are given different colours, (b) when X, Y are given the same colour. *(6 marks)*





5 (i) For each of the graphs shown above, either (a) show that it is planar, or (b) find a subgraph which is a subdivision of $K_{3,3}$ or K_5 and show how it can be drawn on a torus and embedded in a Möbius strip without its edges crossing. (Your drawings must include the labelling of the vertices as given.) *(11 marks)*

(ii) Let G be a simple graph. A graph G' is constructed as follows. Each edge of G corresponds to a vertex of G' and conversely each vertex of G' corresponds to an edge of G. Two vertices of G' are adjacent if and only if the corresponding edges in G have a common end-vertex. What does an edge-colouring of G correspond to in G', and how are $\chi'(G)$ and $\chi(G')$ related? Construct the graph K_4' and verify the connection between $\chi'(K_4)$ and $\chi(K_4')$. *(9 marks)*

State how many vertices the graph K_n' possesses, and show it has exactly $\frac{1}{2}n(n-1)(n-2)$ edges. *(4 marks)*

Is this construction likely to be a useful way to determine the chromatic index of a graph? Give a brief reason for your answer. *(1 mark)*

End of Question Paper