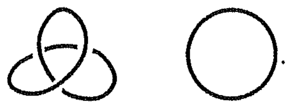
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2010–2011

Knots and Surfaces — 2 hours and 30 minutes

Answer four *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Draw the *Reidemeister moves* and define *Reidemeister equivalence*. Prove that the following diagrams are Reidemeister equivalent:



(8 marks)

(ii) Define the *sign* $\epsilon(c)$ of an oriented crossing c, and define the writhe $w(k)$ of an oriented link diagram k. Show that the writhe is not invariant under the Reidemeister moves. *(5 marks)*

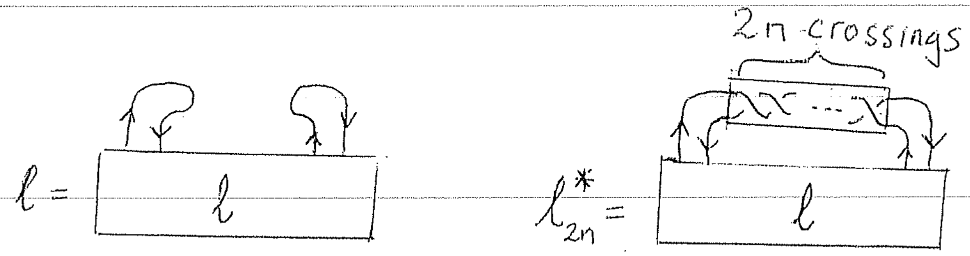
(iii) (a) For an oriented link diagram k the *total linking number* $\mathrm{lk}(k)$ is defined as follows:

$$\mathrm{lk}(k) := \frac{1}{2}\sum_{c} \epsilon(c),$$

where the sum is taken over all crossings c involving two *different* components and where $\epsilon(c)$ denotes the sign of the crossing c. Show that this is invariant under the Reidemeister moves. *(8 marks)*

(b) Give two different examples of a two component link which has total linking number zero in which both components are unknotted. [You do not need to prove that they are different.] *(4 marks)*

2 Given a link diagram ℓ, then for $n \geq 0$ we may form a new link diagram ℓ^*_{2n} with $2n$ additional crossings by inserting a string with $2n$ negative twists as illustrated.



(i) (a) Show that for any ℓ if $n \geq 1$ then the Jones polynomial is given by

$$f[\ell^*_{2n}] = B^2 f[\ell^*_{2n-2}] + (B-1)A^2 f[\ell]$$

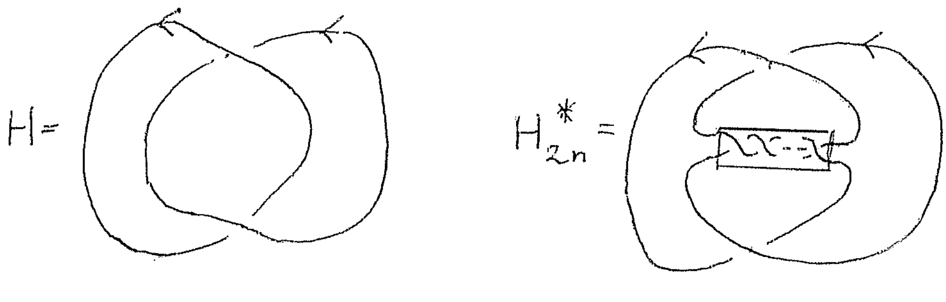
where $B = A^4$. *(8 marks)*

(b) Show by induction on n that for $n \geq 0$ we have the formula

$$f[\ell^*_{2n}] = B^{2n} f[\ell^*_0] + \left\{ \frac{B^{2n}-1}{B+1} \right\} A^2 f[\ell]$$

(6 marks)

(ii) Take $\ell = H$ to be the positive Hopf link, and insert the string of extra crossings in the illustrated band.



(a) Draw H^*_0, H^*_2 and H^*_4 *(3 marks)*

(b) Apply the formula from Part (i) (b) to $\ell = H$ to find $f[H^*_{2n}]$ *(3 marks)*

(c) State a condition satisfied by the Jones polynomial of an amphicheiral link, and hence find the values of n for which H^*_{2n} is amphicheiral. *(5 marks)*

[You may use without proof the fact that $f[H] = -(A^{-10} + A^{-2})$, and that the Jones polynomial satisfies the skein relation

$$A^4 f[K_+] - A^{-4} f[K_-] = (A^{-2} - A^2) f[K_0].]$$

3 (i) (a) What is a *surface word*? Describe how to construct a surface from a surface word. *(5 marks)*

(b) Describe the classification of closed connected surfaces. *(4 marks)*

(ii) Show, using sketches of a plane model, how the word operation which changes a word of the form $AxBCx^{-1}D$ to the form $AyCBy^{-1}D$ does not change the associated surface (here capital letters represent parts of words, B and C non-empty, and lower case letters represent particular edges). *(5 marks)*

(iii) What is an *orientable* surface word? Show that if w is a non-orientable surface word then it is W-equivalent to xxw' where w' is shorter than w. *(5 marks)*

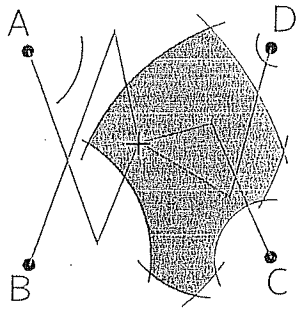
(iv) For each of the following two surface words, use word operations to identify the place of their associated surfaces in the classification.

(a) $abcd^{-1}a^{-1}bc^{-1}d$ *(3 marks)*

(b) $a^{-1}bae^{-1}cdb^{-1}c^{-1}d^{-1}e$ *(3 marks)*

4 (i) Explain what is meant by the *Euler number* of a covering pattern of a compact surface, and use your definition to calculate the Euler number of the standard plane model for the surface $M(g)$, for $g \geq 0$. *(6 marks)*

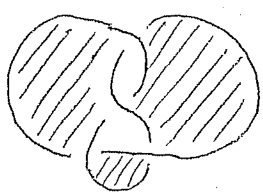
(ii) Consider the linkage of eight rods in the plane illustrated below. The rods can move freely in the plane around the pivots joining them, except that they are anchored to the plane by the pivots at A, B, C and D. The shaded region gives the possible positions of the pivot marked with a "+".



Find the Euler characteristic of the phase space of the linkage, and hence find the genus of the corresponding surface (you may assume it is orientable). *(7 marks)*

(iii) (a) Give a formula for the Euler characteristic $\chi(M \cup N)$ in terms of $\chi(M), \chi(N)$ and $\chi(M \cap N)$ where M and N are surfaces with boundary, stuck together along $M \cap N$. (You may assume that M and N have compatible covering patterns.) *(2 marks)*

(b) A link diagram k divides the plane into a number of regions. You may assume that the regions have been coloured black and white in such a way that the colours on the two sides of any edge are of different colours and the infinite region is white. For example, here is a picture of the Figure Eight diagram coloured in this way.



Now form a surface $\Sigma(k)$ with boundary k from the black regions just as it appears in the picture, by interpreting each crossing as a half-twisted strip. If there are b black regions and c crossings, and if the link itself has n vertices, find the Euler characteristic of $\Sigma(k)$. Now suppose k has d components, and form a closed surface $\widehat{\Sigma}(k)$ by sticking on a new black disc to each component of k. Find the genus in terms of $\widehat{\Sigma}(k)$ in terms of b, c and d on the assumption that it is orientable. Give a criterion based on the diagram which guarantees that $\widehat{\Sigma}(k)$ is orientable.

Draw a diagram of the trefoil knot which gives an orientable surface and find its genus. *(10 marks)*

5 Are the following statements true or false? Justify your answers by giving a proof if they are true or by using an appropriate example if they are false.

(i) The trefoil knot is amphicheiral. *(5 marks)*

(ii) Any link diagram can be changed into an unlink diagram by altering some of its crossings. *(5 marks)*

(iii) The edges of a regular 16-gon may be glued in pairs to obtain a surface of genus 5. *(5 marks)*

(iv) The Jones polynomial of a knot evaluated at $A = 1$ is never even. *(5 marks)*

(v) Any orientable surface can be changed into the sphere by cutting it along a number of disjoint circles and regluing appropriately, using all the pieces. *(5 marks)*

End of Question Paper