The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

MAS346 Groups and Symmetry 2 **hours and** 30 **minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 Let G and H be groups and let $f : G \to H$ be a function.

(i) Explain what it means to say that

(a) f is a group epimorphism, ***(2 marks)***

(b) f is a group monomorphism. ***(2 marks)***

(ii) Prove that the composition $a \circ b$ of two group isomorphisms $a, b : G \to G$ is a group isomorphism. ***(4 marks)***

(iii) Let G be a group such that $g^2 = e$ for all elements $g \in G$, where e is the identity of G. Prove that G is an abelian group. ***(4 marks)***

(iv) For the rest of the question you can assume the following properties of the determinants of 2×2 matrices:
$\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$, $\det(E) = 1$.

(a) Define the group O_2, and the elements R_θ and S_θ of O_2. ***(3 marks)***

(b) Prove that $\det : O_2 \to \{\pm 1\}$ is an epimorphism. ***(6 marks)***

(c) Define the subgroup SO_2 of O_2 and prove that it is a normal subgroup of O_2. ***(4 marks)***

2 (i) Define the centre of a group G and prove that it is a normal subgroup. ***(5 marks)***

(ii) Find the centre of the group $\mathrm{GL}_3(\mathbb{R})$. ***(7 marks)***

(iii) Let H be the set of all 3×3 real matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \text{ with } x, y, z \in \mathbb{R}.$$

(a) Show that H is a subgroup of $\mathrm{GL}_3(\mathbb{R})$. ***(5 marks)***

(b) Find the centre $Z(H)$ of H and show that the quotient group $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ with vector addition. (Hint: Use the first isomorphism theorem.) ***(8 marks)***

3 (i) Give a definition of the direct product $G_1 \times \cdots \times G_n$ of groups $G_1, \ldots, G_n$. Your answer should include a verification of the axioms of a group. ***(4 marks)***

(ii) Prove that a group G is the direct product of its subgroups $G_1, \ldots, G_n$ if and only if

(a) $G = G_1 \cdots G_n$ and $G_i \triangleleft G$ for $i = 1, \ldots, n$, and

(b) for each $i = 1, \ldots, n$, $G_i \cap (G_1 \cdots G_{i-1} G_{i+1} \cdots G_n) = \{e\}$ where e is the identity of the group G.

(Hint: You may assume that G is isomorphic to $G_1 \times G_2 \times \ldots \times G_n$ if every element $a \in G$ is a unique product $a_1 \cdots a_n$ of elements $a_i \in G_i$ for $i = 1, \ldots, n$ and $G_i \triangleleft G$ for $i = 1, \ldots, n$.) ***(9 marks)***

(iii) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group with multiplication given by the rules:

$$i^2 = j^2 = k^2 = -1, \ (\pm 1)a = a(\pm 1) = \pm a \ \text{ for all } a \in Q,$$

$$ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

Find all the subgroups of Q and justify your response. ***(8 marks)***

(iv) Is the quaternion group Q a direct product of any two of its nontrivial subgroups? Justify your answer. ***(4 marks)***

4 (i) Let X be a subset of $\mathbb{R}^n$.

(a) Define the symmetry group $\mathrm{Symm}(X)$ and the direct symmetry group $\mathrm{Dir}(X)$. ***(4 marks)***

(b) Prove that for any $X \subset \mathbb{R}^n$ and $A \in O_n$ one has

$$\mathrm{Dir}(AX) = A\mathrm{Dir}(X)A^{-1}.$$

(6 marks)

(ii) Let C be the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$.

(a) Let M_1, M_2 and M_3 be the x-, y-, and z-axis. Explain how to use these to define a homomorphism $\Psi : \mathrm{Dir}(C) \to S_3$. ***(3 marks)***

(b) By describing some elements $g \in \mathrm{Dir}(C)$ and the corresponding permutations $\Psi(g)$, show that Ψ is surjective. ***(6 marks)***

(c) Let G be the group of matrices of the form

$$g = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

such that $\epsilon_i \in \{\pm 1\}$ and $\epsilon_1\epsilon_2\epsilon_3 = 1$. Show that $G = \ker \Psi$. ***(6 marks)***

5 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Let G be a group of order 175.

(a) How many Sylow 5-subgroups does G have? Justify your answer. ***(2 marks)***

(b) Show that G has a normal subgroup N of order 7. ***(5 marks)***

(c) Show that if P is a Sylow 5-subgroup of G then every element of P commutes with every element of N. ***(7 marks)***

(d) Define the function $\phi : P \times N \to G$ by $\phi(x, y) = xy$. Prove that ϕ is a monomorphism and show that $G \simeq P \times N$. ***(6 marks)***

End of Question Paper