



The
University
Of
Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn 2010-2011

Financial Mathematics

2 hours and 30 minutes

*Answer **three** questions. If you answer more than three questions only your best three will be counted.*

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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- 1 (i) Assume that the risk-free spot interest rate for all maturities is 3%. Consider a risk-free perpetual bond which pays £1 once a year for perpetuity. Assume that the first annual payment is in 6 months.

(a) Find the price of the bond. *(7 marks)*

(b) Explain why the price of this bond is a decreasing function of the risk-free spot interest rates. *(2 marks)*

- (ii) Consider the following two risk-free zero-coupon bonds with face value of £100:

Time to maturity (in years)	Bond price (in £)
1	97.53
3	90.03

Suppose that you are offered by a risk-free institution the opportunity to deposit or borrow £1,000,000 in one year for a period of two years earning an interest rate of 3%. Describe in detail an arbitrage opportunity available to you. *(13 marks)*

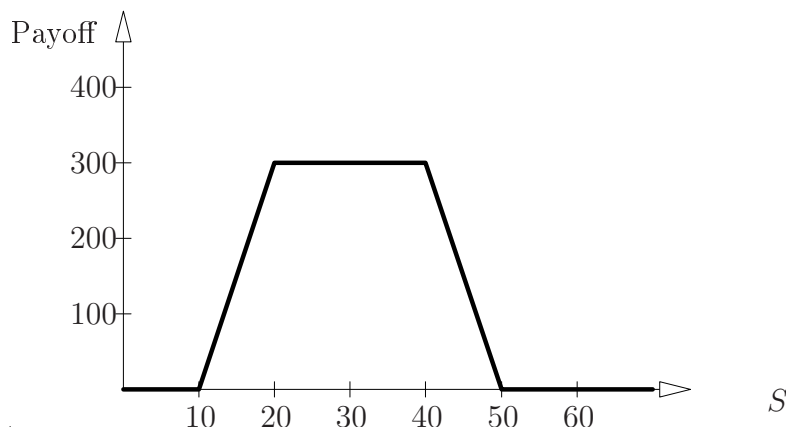
- (iii) Consider a forward contract on a foreign currency. By considering two portfolios involving positions on the forward contract and domestic and foreign currency deposits show that

$$Fe^{-rT} = Se^{-r_f T},$$

where F is the forward exchange rate (i.e., pounds paid for 1 unit of foreign currency on delivery), S is the spot price of the asset (i.e., pounds paid at the present for 1 unit of foreign currency), T is the time to maturity of the forward contract (in years), r is the T -year spot interest rate for domestic currency and r_f is the T -year spot interest rate for the foreign currency.

(11 marks)

- 2 (i) (a) Describe a portfolio consisting entirely of European options on a stock with different strike prices and the same expiration time $T > 0$, and whose payoff at time T as a function of S , the spot price of the stock at time T , is described by the graph below. *(4 marks)*



- (b) Use the properties of payoff function of the portfolio you constructed in (a) to deduce an inequality satisfied by the spot prices of the components of that portfolio. *(5 marks)*
- (ii) Consider an American call option on shares with strike price X and expiring in T years. Assume that owning the shares for the next T years does not entitle the owner to dividends. Let $0 \leq \tau < T$, and let S_τ denote the share price at time τ .
- (a) What is the payoff obtained by exercising the option at time τ ? *(3 marks)*
- (b) Explain how (a) implies that the option should not be exercised at time τ . *(5 marks)*
- (c) Deduce that the price of the option equals that of a European call option with the same underlying asset, strike price and expiration date. *(2 marks)*
- (iii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 3%.
- (a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £22. *(11 marks)*
- (b) Describe all circumstances in which a rational investor should exercise the option. *(3 marks)*

3 (i) Explain the principle of risk-neutral valuation. **(4 marks)**

(ii) Consider a derivative on a stock which entitles the holder to one payoff at time T ; the amount of this payoff is £1 if the stock price S_T at time T is at most a , for some positive number a , and zero otherwise. Let S be the price of the stock and assume, as usual, that S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and $\sigma > 0$ and where B is a Brownian motion. Assume further that all interest rates are constant and equal to r .

(a) Use Ito's Lemma to show that $\log S$ follows the process

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB. \quad \textbf{(6 marks)}$$

(b) Find an expression for the probability *in a risk-neutral world* of the event $S_T \leq a$. **(8 marks)**

(c) Apply a risk-neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative equals

$$e^{-r(T-t)} N \left(\frac{\log(S_t/a) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right),$$

where N is the cumulative distribution function of the standard normal distribution. **(3 marks)**

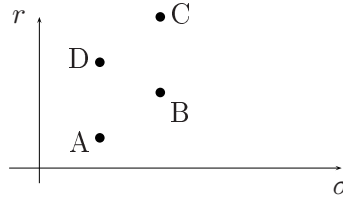
(iii) (a) Show that $f(S, t) = e^{(4r+\sigma^2)t/8} \sqrt{S}$ is a solution of the Black-Scholes PDE

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \quad \textbf{(5 marks)}$$

(b) Deduce that the function $v(S, t) = e^{(4r+\sigma^2)(t-T)/8} \sqrt{S}$ is also a solution of the Black-Scholes PDE. **(2 marks)**

(c) Consider a derivative on a certain stock (whose price S , as always, follows the process $dS = S\mu dt + S\sigma dB$) which provides a single payoff at time $T > 0$ amounting to $\sqrt{S_T}$. Find the value of the derivative at time $0 \leq t \leq T$. **(5 marks)**

- 4 (i) Explain the following terms in the context of portfolio theory:
- (a) *market portfolio*, (2 marks)
 - (b) *beta coefficient of an investment*, (2 marks)
 - (c) *security market line*, (2 marks)
 - (d) *feasible set*, and (2 marks)
 - (e) *efficient frontier*. (2 marks)
- (ii) Sketch the following:
- (a) an example of a feasible set and efficient frontier in the absence of a risk-free investment, (2 marks)
 - (b) an example of a feasible set, market portfolio and efficient frontier in a market containing a risk-free investment. (3 marks)
- (iii) Which portfolios among A,B,C and D pictured below could be on an efficient frontier? Explain your answer in detail. (2 marks)



- (iv) Consider a world where there are only two risky investments: *Guns R Us* and *Yummy Butter* stocks.

	Number of shares	Price per share	Expected return	Standard deviation of return
Guns R Us	5,000,000	£1	20%	30%
Yummy Butter	3,000,000	£5	10%	10%

The correlation between the returns of these two stocks is $1/10$.

- (a) What is the market portfolio? (2 marks)
- (b) What are the expected return and standard deviation of returns of the market portfolio? (3 marks)
- (c) Find the beta-coefficient of *Guns R Us*. (3 marks)
- (d) What is the risk-free return in this world? (3 marks)
- (e) Assume that risk-free deposits are available. Of all portfolios consisting of cash-deposits and the two risky investments with expected returns of 10%, which one has the lowest standard deviation of returns. (5 marks)

End of Question Paper