The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Time Series **2 hours**

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 99 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) (a) In the context of descriptive analysis of time series x_t, briefly explain why a moving average for even span s is *not* defined as

$$\frac{1}{s}(x_{t-s/2} + x_{t-s/2-1} + \cdots + x_{t-1} + x_t + x_{t+1} + \cdots + x_{t+s/2-1}).$$

(3 marks)

(b) Consider the time series with values

$$x_1 = 5, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 5, \quad x_5 = 7, \quad x_6 = 6, \quad x_7 = 3.$$

Using the *correct* definition of the even-span moving average, calculate moving averages of span 4, at the time points 3, 4 and 5. ***(5 marks)***

(ii) A time series of length 70 gave values for the sample autocorrelation function (ACF), denoted by r_h and values for the partial ACF, denoted by a_h, according to the table below.

Lag h	1	2	3	4
r_h	0.58	0.43	0.37	0.22
a_h	*	*	0.19	0.21

(a) Using this table, find the values of a_1 and a_2, indicated in the table by stars. ***(8 marks)***

(b) Test whether this time series is consistent with a white noise process, a moving average model and an autoregressive model. ***(13 marks)***

(c) Suggest a model which you would expect to fit well to this time series data. ***(4 marks)***

2 Consider the time series model

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t + \frac{1}{3}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2}, \tag{1}$$

where ϵ_t is a white noise process with variance 3, i.e. $\epsilon_t \sim WN(0, 3)$.

(i) Give the abbreviated name of the model for X_t. ***(2 marks)***

(ii) Write down model (1) in compact form, using the backward shift operator B. ***(4 marks)***

(iii) Show that model (1) is causal and invertible. ***(9 marks)***

(iv) Find the variance of X_t. ***(18 marks)***

3 (i) Suppose that X_t follows the model $SARIMA(0,1,2) \times (1,0,0)_{12}$.

(a) Write down this model for X_t in analytical form, i.e. write down X_t in terms of X_{t-j} etc, for $j = 1, 2, \ldots$. ***(4 marks)***

(b) State what kind of time series data the above model is most likely to be suitable for? ***(1 mark)***

(ii) Consider the model

$$X_t = 0.2X_{t-3} + \epsilon_t - 0.4\epsilon_{t-1},$$

where ϵ_t is a white noise process, i.e. $\epsilon_t \sim WN(0,1)$. This model is fitted to the following data set:

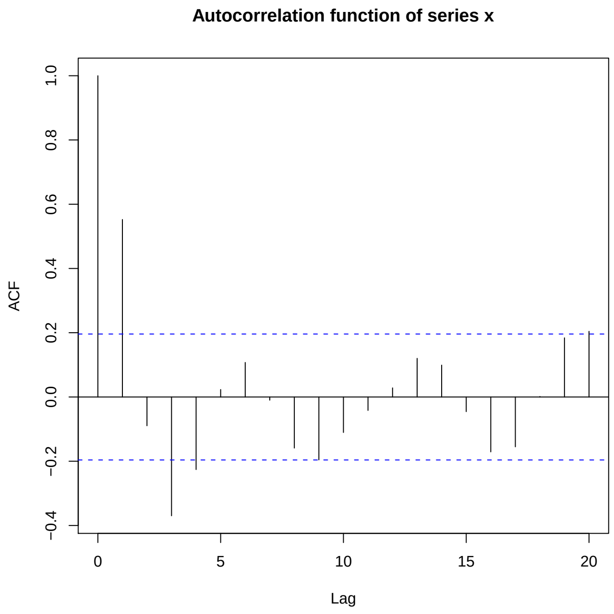
Time t	1	2	3	4	5	6
x_t	6	1	-2	7	2	-2

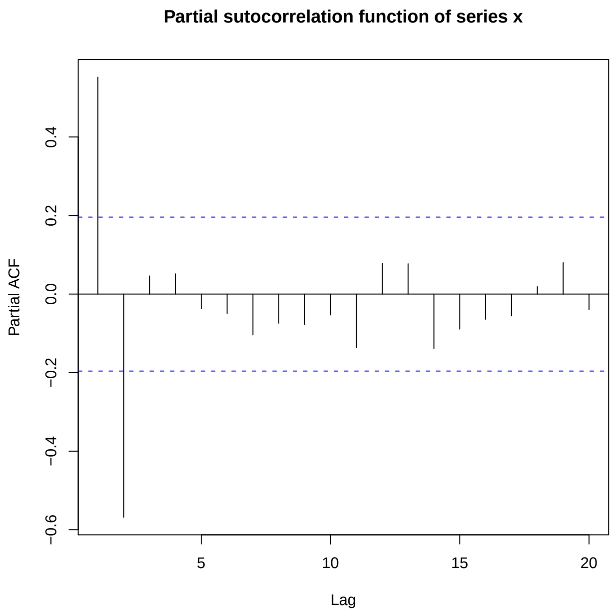
Calculate the one-step ahead and two-step ahead forecasts of the value at $t = 6$ (i.e. the one-step ahead forecast using data up to time $t = 5$ and the two-step forecast using data up to time $t = 4$) and give their actual respective forecast errors. From this information only, comment on the quality of forecasting. ***(14 marks)***

(iii) In a time series of length 60, we fit an ARMA(1,3) model and the residual autocorrelations at lags $1, 2, 3, 4, 5$ are 0.3, 0.1, 0.02, 0.004, 0.002, respectively. If the cumulative distribution function of the chi-square distribution with 1 degree of freedom at point 6.343 is 0.988, then carry out the modified Box-Pierce (Ljung-Box-Pierce) test, in order to test the adequacy of the model fit. Clearly state the null hypothesis and comment on the strength of the evidence of the test and on the overall model adequacy, based on this test. ***(6 marks)***

(iv) A time series x_t gave values of the sample ACF (autocorrelation function) and sample partial ACF, as shown in the two plots overleaf. Based on this information only, determine whether the time series is weakly stationary or not and suggest an ARIMA model which you would expect to be a good fit to this time series. ***(8 marks)***

 Question 3 continued on next page


Autocorrelation function of series x
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15
20
Lag


Partial sutocorrelation function of series x
Partial ACF
0.4
0.2
0.0
−0.2
−0.4
−0.6
5
10
15
20
Lag

3 (continued)

4 Consider the trend dynamic linear model, given by equations

$$X_t = [1, 0] \begin{bmatrix} \theta_{1t} \\ \theta_{2t} \end{bmatrix} + \epsilon_t = \boldsymbol{F}^T \boldsymbol{\theta}_t + \epsilon_t, \tag{2}$$

$$\boldsymbol{\theta}_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t = \boldsymbol{G}\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \tag{3}$$

where $\boldsymbol{\theta}_t = [\theta_{1t}, \theta_{2t}]^T$ is a state vector, ϵ_t follows a normal distribution with zero mean and variance 50, and $\boldsymbol{\omega}_t$ follows a bivariate normal distribution with zero mean vector and covariance matrix

$$\boldsymbol{W} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

written as

$$\boldsymbol{\omega}_t \sim N_2 \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \right\}.$$

It is also assumed that ϵ_t and $\boldsymbol{\omega}_t$ are mutually and individually independent, and they are independent of the initial state $\boldsymbol{\theta}_0$. Suppose that $x_1, x_2, \ldots, x_n$ values of the time series are observed and that the posterior distribution of $\boldsymbol{\theta}_n$, given information $x^n = (x_1, \ldots, x_n)$ is given by

$$\boldsymbol{\theta}_n | x^n \sim N_2 \left\{ \begin{bmatrix} 250 \\ 100 \end{bmatrix}, \begin{bmatrix} 10 & 0 \\ 0 & 33 \end{bmatrix} \right\}$$

For some positive integer $k > 0$, define the new time series

$$S_n = X_{n+1} + X_{n+2} + \cdots + X_{n+k}$$

(i) Show that the k-step forecast function of $\{X_t\}$ is $f_n(k) = E(X_{t+n}|x^n) = 100k + 250$. ***(8 marks)***

(ii) Find the posterior mean of S_n, given x^n, for $k = 2$. ***(5 marks)***

(iii) For $k = 2$, show that, given x^n, the covariance of X_{n+1} and X_{n+2} is 96, and hence calculate the posterior variance of S_n, given x^n. ***(18 marks)***

(iv) Derive the posterior distribution of S_n, given x^n, for $k = 2$. ***(2 marks)***

End of Question Paper