The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2010–2011

Topics in Advanced Fluid Mechanics — **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Starting from the 3D Euler equations for an incompressible fluid of a constant unit density

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p,$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

derive the following set of equations for the impulse defined by $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$,

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla\boldsymbol{u})^T\boldsymbol{\gamma} + \nabla\lambda,$$

$$\frac{D\phi}{Dt} = p - \frac{|\boldsymbol{u}|^2}{2} + \lambda,$$

where λ is an arbitrary scalar function of $\boldsymbol{x}$ and t. ***(19 marks)***

(ii) By using the impulse equations for a special choice of $\lambda = 0$ (geometric gauge) and the vorticity equations

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u},$$

show that $\boldsymbol{\gamma}\cdot\boldsymbol{\omega}$ is conserved

$$\frac{D}{Dt}(\boldsymbol{\gamma}\cdot\boldsymbol{\omega}) = 0.$$

(6 marks)

2 (i) We consider Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \qquad (A)$$

with an initial condition $u(x,0) = u_0(x)$. By following the steps below, reduce (A) to

$$\frac{\partial \psi}{\partial t} = \nu\frac{\partial^2 \psi}{\partial x^2}, \qquad (B)$$

by a transformation

$$u = -2\nu(\log\psi)_x,$$

where subscripts denote differentiation.

(1) Show that $\quad u_t = -2\nu\left(\frac{\psi_t}{\psi}\right)_x.$ ***(3 marks)***

(2) Show that $\quad \nu u_x - \frac{u^2}{2} = -2\nu^2\frac{\psi_{xx}}{\psi}.$ ***(4 marks)***

(3) By writing (A) as $u_t = \left(\nu u_x - \frac{u^2}{2}\right)_x$, show that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = -2\nu\left(\frac{\psi_t - \nu\psi_{xx}}{\psi}\right)_x.$$

(3 marks)

(ii) We consider steady Burgers equation in 1D

$$u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

under boundary conditions

$$\lim_{x\to-\infty} u(x) = u_1, \lim_{x\to\infty} u(x) = u_2,$$

where $u_1 > 0$.

(1) Solve this equation for $u(x)$. Show that we must have $u_2 = -u_1$ and that

$$u(x) = -u_1\tanh\left(\frac{u_1}{2\nu}x + c\right),$$

where c is a constant. (Hint: $\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$.)

(9 marks)

(2) Prove that the total energy dissipation rate

$$\epsilon = \nu\int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

is independent of ν. $\left(\text{Hint : } \int_{-\infty}^{\infty}\frac{dX}{\cosh^4 X} = \frac{4}{3}.\right)$ ***(6 marks)***

3 Consider a model equation for vorticity defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega],$$

with an initial condition

$$\omega(x, t = 0) = \omega_0(x).$$

Here $H[\omega]$ denotes a Hilbert transform

$$H[\omega](x) = \frac{1}{\pi} \text{PV} \int \frac{\omega(y)}{x - y} dy,$$

where PV means a principal value.

(1) Derive an equation for $H[\omega]$ as

$$\frac{\partial H[\omega]}{\partial t} = \frac{1}{2} \left(H[\omega]^2 - \omega^2 \right).$$

(Hint: $H[fg] = H[f]g + fH[g] + H[H[f]H[g]]$, $H[H[f]] = -f$ for any f, g.)

(7 marks)

(2) Drive an equation for $f = \omega + iH[\omega]$ as follows

$$\frac{\partial f(x,t)}{\partial t} = -\frac{i}{2} f(x,t)^2.$$

(4 marks)

(3) By solving the above equation for $f(x, t)$, obtain expressions for an exact solution as

$$\omega(x,t) = \frac{\omega_0(x)}{(1 - \frac{t}{2} H[\omega_0])^2 + (\frac{t}{2}\omega_0(x))^2}$$

and

$$H[\omega](x,t) = \frac{H[\omega_0](1 - \frac{t}{2} H[\omega_0]) - \frac{t}{2}\omega_0(x)^2}{(1 - \frac{t}{2} H[\omega_0])^2 + (\frac{t}{2}\omega_0(x))^2}.$$

(7 marks)

(4) Consider simple initial data $\omega_0(x) = \sin x$ (hence, $H[\omega_0](x) = -\cos x$). By using the explicit solutions above, show that

$$\omega(x,t) = \frac{\sin x}{1 + t\cos x + (t/2)^2}$$

and that

$$\max_{0 \le x \le 2\pi} \omega(x,t) = \frac{4}{4 - t^2}.$$

(7 marks)

4 We consider a system of N point-vortices:

$$\frac{dx_i}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2},$$

$$\frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2},$$

where $\sum{}'$ denotes a summation excluding $j = i$. Here (x_i, y_i), $i = 1, 2, \dots, N$ denotes coordinates of a point vortex of strength κ_i.

(1) Show that

$$H = -\frac{1}{8\pi} \sum_{j,k=1}^{N}{}' \kappa_j \kappa_k \log\left((x_j - x_k)^2 + (y_j - y_k)^2\right)$$

is a constant of motion, by rewriting the above set of equations as

$$\kappa_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \ \kappa_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}, \ \text{(no summation)}.$$

(7 marks)

(2) Show that

$$\sum_{i=1}^{N} \kappa_i x_i, \ \sum_{i=1}^{N} \kappa_i y_i, \ \sum_{i=1}^{N} \kappa_i (x_i^2 + y_i^2)$$

are constants of motion. ***(9 marks)***

(3) Determine the motion for the case of two point vortices $N = 2$, $\kappa_1 = -\kappa_2 (= \kappa > 0)$. ***(9 marks)***

5 The motion of a vortex layer of uniform strength is governed by

$$\frac{\partial z(\alpha,t)^*}{\partial t} = -\frac{\mathrm{i}}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{z(\alpha,t)-z(\beta,t)}d\beta, \qquad (A)$$

where

$$z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$$

denotes the position of a fluid particle α on the layer, $*$ complex conjugate and PV a principal value. We study its stability property of a flat state $z_0(\alpha) = \alpha$.

(1) By setting $z(\alpha) = \alpha + if(\alpha,t)$, where $f(\alpha,0) = 0$, derive from (A)

$$\frac{\partial f(\alpha)^*}{\partial t} = \frac{1}{2\pi}\mathrm{PV}\int \frac{d\beta}{\alpha-\beta+i(f(\alpha)-f(\beta))}. \qquad (B)$$

(3 marks)

(2) Using a series expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ on the right-hand side of (B), derive an approximate equation

$$\frac{\partial f(\alpha)^*}{\partial t} = \frac{-i}{2\pi}\mathrm{PV}\int \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)\frac{d\beta}{\alpha-\beta}. \qquad (C)$$

(Hint: $H[\mathrm{const}] = 0$, where H is a Hilbert transform; see Q3 for its definition.)

(6 marks)

(3) Using the following relation

$$\mathrm{PV}\int \frac{f(\alpha)-f(\beta)}{(\alpha-\beta)^2}d\beta = -\mathrm{PV}\int \frac{1}{\alpha-\beta}\frac{\partial}{\partial\beta}\left(f(\alpha)-f(\beta)\right)d\beta,$$

show that we can write (C) as

$$\frac{\partial f^*}{\partial t} = -\frac{i}{2}H\left[\frac{\partial f}{\partial \alpha}\right].$$

(4 marks)

(4) Setting $f(\alpha,t) = p(\alpha,t) + iq(\alpha,t)$ with real functions p, q, derive

$$\frac{\partial^2 p}{\partial t^2} = -\frac{1}{4}\frac{\partial^2 p}{\partial \alpha^2}$$

(Hint: $H[H[f]] = -f$, $\frac{\partial}{\partial\alpha}H[f] = H\left[\frac{\partial f}{\partial\alpha}\right]$, for any $f(x)$.) ***(6 marks)***

(5) Using a Fourier series $p(\alpha,t) = \sum_{k=-\infty}^{\infty}\tilde{p}(k,t)\exp(ik\alpha)$ to derive

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = \left(\frac{k}{2}\right)^2\tilde{p}.$$

Solve the above equation to determine its linear stability. ***(6 marks)***

End of Question Paper