The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2010-2011

Analytical Dynamics and Classical Field Theory **3 Hours**

Marks will be awarded for your best FIVE answers. You are advised not to attempt more than FIVE questions.

1 The Lagrange–function of a system with N degrees of freedom $L(q_i, \dot{q}_i, t) = T - V$ is a function of the generalized coordinates q_i, their time-derivatives $\dot{q}_i = dq_i/dt$ (also called the generalized velocities) and time t. The quantity T is the total kinetic energy and V is the potential energy.

(i) What does the number of generalized coordinates count? Are the generalized coordinates independent of each other? ***(2 marks)***

(ii) The Euler–Lagrange equations read:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Given the Lagrange–function L, define the action. State Hamilton's principle and derive the Euler–Lagrange equations from Hamilton's principle. ***(13 marks)***

(iii) Deduce momentum conservation for a free particle from the Euler–Lagrange equations. ***(5 marks)***

2 A system of N particles is described by a Lagrange–function $L(q_i, \dot{q}_i, t)$, with $i = 1, \ldots, N$.

(i) Define the canonical momenta, the Hamilton function and state Hamilton's equations. ***(5 marks)***

(ii) Consider two functions f and g, which are functions of q_i, the canonical momenta P_i and time t. Write down the definition of the Poisson bracket $[f, g]$ between f and g. Show that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]\,.$$

(4 marks)

(iii) The angular momentum is defined as $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector of a particle and $\vec{p}$ is its momentum. Show that the Poisson bracket between L_x and L_y satisfies

$$[L_x, L_y] = L_z,$$

where $\vec{L} = (L_x, L_y, L_z)$. ***(11 marks)***

3 (i) A system of N particles with masses m_i and position vectors $\vec{r}_i$ move under applied forces $\vec{F}_i$ such that $\vec{r}_i$ and $\dot{\vec{r}}_i$ remain finite. Define the total kinetic energy T of the system and show that the quantity

$$W = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i$$

satisfies $2\bar{T} + \bar{W} = 0$, where the bar denotes the time average

$$\bar{f} = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(t')dt'.$$

Show further that for a single particle moving in a central force field of potential $V = ar^2$,

$$\bar{T} = \bar{V}.$$

(10 marks)

(ii) A particle of mass m moves along the x axis under a conservative force field with potential energy

$$V = \frac{ax}{b^2 + x^2},$$

where a and b are positive constants. Show that the position of stable equilibrium is at $x = -b$ and that the period of small oscillations around $x = -b$ is $2\pi\sqrt{2mb^3/a}$. [Hint: The equation of motion can be found from the energy conservation equation $T + V =$ constant: take the time-derivative of that equation and write $x = -b + \delta x$ with $|\delta x| \ll b$ to find an equation for δx to linear order.] ***(10 marks)***

4 Lorentz–transformations describe the transformation of the coordinates between two inertial systems moving with relative velocity $\vec{v}$ and are represented by a matrix $\Lambda^{\alpha}{}_{\beta}$.

(i) How does a contravariant Lorentz-vector a^{μ} transform under Lorentz–transformations? Define the covariant Lorentz-vector a_{μ}. How does the tensor $\sigma^{\alpha\beta\gamma}$ transform under Lorentz–transformations? ***(4 marks)***

(ii) Given that the tensor $\sigma^{\mu\nu\rho} = A^{\mu}B^{\nu}C^{\rho}$ transforms as a four–tensor and A^{μ} and C^{ρ} transform as four-vectors (with A^{μ} and C^{ρ} non-zero), show that B^{ν} transforms as a four–vector as well. ***(6 marks)***

(iii) The four-velocity of a particle is the vector $u^{\mu} = dx^{\mu}/d\tau$, where $x^{\mu}(\tau) = (ct, x, y, z)$ is the four-vector describing the motion of the particle and τ is the proper time and c is the speed of light. Give the definition of the proper time τ and show that $u^{\mu}u_{\mu} = c^2$. ***(4 marks)***

(iv) A particle is moving under the influence of an electromagnetic field, described by the field strength tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. The equation of motion is

$$m\frac{du^{\mu}}{d\tau} = \frac{q}{c}F^{\mu\nu}u_{\nu},$$

where m is the mass of the particle and q its charge. Use the equation of motion to show that if $u^{\mu}u_{\mu} = c^2$ at $\tau = 0$, then $u^{\mu}u_{\mu} = c^2$ for all τ. ***(6 marks)***

5 Consider the following Lagrangian $\mathcal{L}$, describing a vector field A^{μ} with mass m ($m \geq 0$):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_{\mu}A^{\mu},$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

(i) Consider the transformation $A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\theta$, where θ is a function of the coordinates x^{ν}. Show that the Lagrangian is invariant under these transformations if $m = 0$. ***(5 marks)***

(ii) Show that the Euler–Lagrange equations for the Lagrangian above lead to

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}A^{\alpha} + m^2 A^{\alpha} = 0,$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor in Minkowski spacetime.

(15 marks)

6 State Noether's theorem. ***(2 marks)***

Consider the following action, describing a scalar field:

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) \right).$$

You are given the following three different potentials:

- $V(\phi) = 0$,
- $V(\phi) = \frac{1}{2} m^2 \phi^2$ and
- $V(\phi) = \frac{1}{4} \lambda \phi^4$.

Additionally, consider the following transformations with two parameters α and d_ϕ of the form

$$\begin{aligned} x^\mu &\to e^\alpha x^\mu \\ \phi(x) &\to \phi(x) \exp(-d_\phi \alpha), \end{aligned} \tag{1}$$

where both α and d_ϕ are constants. For which d_ϕ and $V(\phi)$ are the transformations a symmetry of the action above? ***(12 marks)***

For the case(s) in which the transformations above are a symmetry transformation, find the associated Noether current. You are given that the Noether current can be written as

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left[A^\nu(x) \partial_\nu \phi - F(\phi, \partial \phi) \right] - A^\mu(x) \mathcal{L},$$

for general infinitesimal transformations of the form (ϵ is a small parameter)

$$\begin{aligned} x'^\mu &= x^\mu + \epsilon A^\mu(x) \\ \phi'(x') &= \phi(x) + \epsilon F(\phi, \partial \phi) \end{aligned} \tag{2}$$

Hint: Assume that α is small and expand the transformations (1) to bring them into the form (2). ***(6 marks)***

End of Question Paper