Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010–11

MAS420 Signal Processing — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) You are given that the set $\phi_n(t) = e^{in\sigma t} : -\infty < n < \infty$, where $\sigma = \dfrac{2\pi}{T}$, forms an orthonormal basis for the Hilbert space of finite power signals of period T with inner product

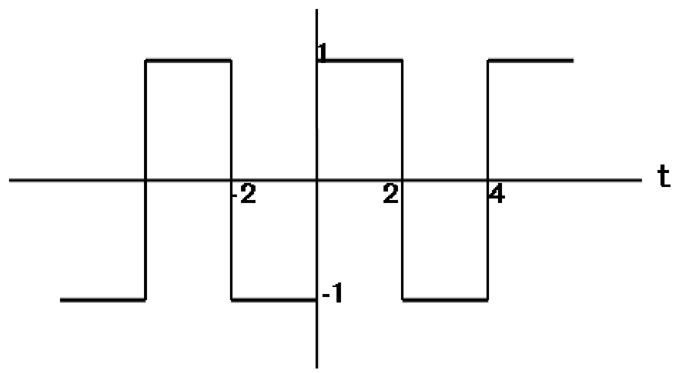
$$(f, g) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t)dt.$$

Prove Parseval's theorem

$$||f||^2 = \sum_n |c_n|^2 \text{ where } c_n = (f, \phi_n).$$

(4 marks)

(ii) Write down the period and the fundamental frequency in rad/s of the periodic signal shown in the figure below and find the complex Fourier coefficients for this signal. ***(7 marks)***



(iii) Use Parseval's theorem and your result in (ii) to derive the equality

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(5 marks)

(iv) The signal is transmitted over a link which will not pass frequencies greater than π rad s^{-1}, but passes other frequencies unchanged. The received signal is $g(t)$. Find an expression for $g(t)$ as a sine/cosine series and show that during transmission approximately 19% of the power is lost.

(9 marks)

2 (i) Prove the convolution theorem

$$f * g(t) \longleftrightarrow F(\omega)G(\omega)$$

where $f(t)$ and $g(t)$ are signals with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively. ***(4 marks)***

(ii) Making use of clear diagrams, evaluate $f * g(t)$ when $f(t) = e^{-\alpha t}U(t)$ and $g(t) = e^{\beta t}U(-t)$, with $\alpha > 0$ and $\beta > 0$. Hence show that

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

. ***(12 marks)***

(iii) The signal

$$f(t) = \frac{1}{\alpha^2 + t^2}$$

is passed through an ideal low-pass filter with system transfer function $H(\omega) = p_\beta(\omega)$. Making use of the duality property of the Fourier transform, find and sketch the amplitude spectrum of the signal, g(t), coming out of the filter. Find the output of the filter if the input is $g(t)\cos\sigma t$ where $\sigma > 2\beta$. ***(9 marks)***

3 (i) If $f(t)$ has a real, positive Fourier transform, $F(\omega)$, prove that $f(0) \geq 0$ and $|f(t)| \leq f(0)$ for any t. ***(4 marks)***

(ii) Define the equivalent rectangle resolution, τ, of a signal, $f(t)$, stating clearly the conditions under which it is defined. Explain, using a clear diagram, how it is related to the signal $f(t)$. ***(5 marks)***

(iii) Using the duality and scaling theorems, show that the signal $f(t) = \dfrac{1}{\alpha - it}$ has Fourier transform

$$F(\omega) = 2\pi e^{-\alpha\omega} U(\omega)$$

(3 marks)

(iv) The signal $f(t) = \dfrac{1}{\alpha - it}$ is passed through an ideal low-pass filter with system transfer function $p_\Omega(\omega)$ to give an output signal $g(t)$. State whether $f(t)$ and/or $g(t)$ are bandlimited. Find the energy in $g(t)$ by a frequency domain calculation and verify that the equivalent rectangle resolution of $g(t)$ is defined.

Making use of the inverse Fourier transform formula, show that

$$g(0) = \frac{1}{\alpha}\left(1 - e^{-\alpha\Omega}\right)$$

and hence find the equivalent rectangle resolution of $g(t)$. ***(10 marks)***

(v) The time-bandwidth theorem states that, for an Ω-bandlimited signal for which τ is defined, $\tau\Omega \geq \pi$. Use this to prove the inequality

$$y\left(\frac{1 + e^{-y}}{1 - e^{-y}}\right) \geq 1, \quad \forall y \geq 0.$$

(3 marks)

4 (i) Define the following:

- a linear shift-invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function, without reference to the STF or convolution.

(4 marks)

(ii) With the aid of a clear diagram, explain what is meant by time domain and frequency domain processing for a LSI system, and why the two approaches are equivalent. ***(3 marks)***

(iii) A system uses integration to smooth noisy signals, i.e. if the input signal is $f(t)$, the output is given by

$$g(t) = \int_{t-T}^{t} f(s)ds$$

where T is a constant. Show that this system is linear and shift-invariant and, using your definitions in part (i), verify that it has STF give by

$$H(\omega) = Te^{-iT\omega/2}\,\mathrm{sinc}(T\omega/2)$$

(7 marks)

(iv) Working directly from your definition in part (i), find the impulse response function, $h(t)$, of the system and verify that $h(t) \longleftrightarrow H(\omega)$. ***(5 marks)***

(v) Use the STF to find the output from the system if the input is

$$f(t) = 1 + \frac{1}{2}\sin\frac{\pi t}{T} + 3\cos\frac{2\pi t}{T},$$

simplifying your answer as much as possible. ***(6 marks)***

5 (i) Assuming the Fourier transform pair $\overline{\delta}_T(t) \longleftrightarrow \sigma\overline{\delta}_\sigma(\omega)$, where $\overline{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$, $\overline{\delta}_\sigma(\omega)$ is defined similarly and $\sigma = 2\pi/T$, prove that

$$f_s(t) \equiv f(t)\overline{\delta}_T(t) \longleftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma),$$

where $F(\omega)$ is the Fourier transform of $f(t)$. Hence show that if $f(t)$ is Ω-bandlimited and $T < \pi/\Omega$, then $f(t)$ can be recovered exactly from the sampled signal $f_s(t)$ by the sinc interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left\{\frac{\sigma}{2}(t-nT)\right\}.$$

Clear diagrams are likely to help your answer. ***(11 marks)***

(ii) Find the Nyquist frequency, in Hz, of the signal

$$f(t) = \mathrm{sinc}^2(t/2).$$ ***(4 marks)***

(iii) This signal is sampled at half the Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(7 marks)***

(iv) Sketch $f(t)$ and $g(t)$ to demonstrate clearly that $f(t) \neq g(t)$. ***(3 marks)***

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-in\sigma t}\,dt$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\,dt$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$