The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

Analytic Number Theory **2 hours 30 minutes**

Answer ***Question 1*** *and three other questions. You are advised* ***not*** *to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

1 (i) State the Prime Number Theorem, and deduce that

$$\pi(x) \sim \frac{x}{\log \pi(x)}.$$

Prove that if p_n denotes the n-th prime, then $p_n \sim n \log n$. ***(9 marks)***

(ii) Let $p_1, \ldots, p_n$ be n distinct primes. For any positive integer x, let $N_n(x)$ be the number of integers between 1 and x with prime divisors only from the set $\{p_1, \ldots, p_n\}$. Show that

$$N_n(x) \leq 2^n \sqrt{x}.$$

Prove that the series $\displaystyle\sum_{\text{primes } p} \frac{1}{p}$ diverges. ***(12 marks)***

(iii) Write down a formula for the highest power of a prime p that divides $n!$ and calculate the number of zeros at the end of 2011! ***(4 marks)***

2 (i) Let $\Phi(X)$ be the polynomial $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$.

(a) Show that the set of primes p for which there is an integer n with $p|\Phi(n)$ is infinite. ***(4 marks)***

(b) Now let a be an integer, and let $p > 7$ be a prime dividing $\Phi(a)$. Show that p divides $a^7 - 1$, and deduce that the residue class of a in $(\mathbb{Z}/p\mathbb{Z})^\times$, the unit group of $\mathbb{Z}/p\mathbb{Z}$, has order 7. ***(6 marks)***

(c) Now deduce that $7|p-1$, and conclude that there are infinitely many primes of the form $7n + 1$. ***(4 marks)***

(ii) State Bertrand's Postulate, and deduce from it that there are infinitely many prime numbers beginning with the digit 1. ***(5 marks)***

Also deduce from it that $\sum_{k=1}^{n} \frac{(-1)^k}{2k+1}$ is never an integer for $n \geq 1$.

(6 marks)

3 (i) Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be an arithmetic function, and let

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

What can you say about the analyticity of $F(s)$ if the sequence $\left(\sum_{k=1}^{n} f(k)\right)_{n=1}^{\infty}$ is bounded? Deduce that if $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_0}}$ converges at $s_0 \in \mathbb{C}$, then $F(s)$ is analytic in the half-plane $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. ***(5 marks)***

(ii) Recall that the *Riemann zeta function* $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Write down the *Euler product* for $\zeta(s)$, indicating clearly in what region of the complex plane the formula is valid. ***(2 marks)***

Derive a relation between $\zeta(s)$ and the series

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots,$$

and explain how you could extend the definition of $\zeta(s)$ to $\mathrm{Re}(s) > 0$.
(7 marks)

(iii) Let $f, g, h : \mathbb{N} \longrightarrow \mathbb{C}$ be three arithmetic functions related by the following formal product of Dirichlet series:

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

Write down the relation between f, g and h. Write down the (formal) Euler product expansion of $\sum \frac{f(n)}{n^s}$ if f is a multiplicative arithmetic function.
(4 marks)

For $\mathrm{Re}(s) > 2$, show that

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$

Here, $\phi(n)$ is the Euler ϕ-function. You can freely use the fact that ϕ is multiplicative and that $\phi(p^n) = p^n - p^{n-1}$ for p a prime, $n \geq 1$. ***(7 marks)***

4 (i) Recall that the Bernoulli polynomials $B_n(x)$ for $n \geq 0$ are defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

(a) Show that $\frac{d}{dx} B_n(x) = nB_{n-1}(x)$. ***(3 marks)***

(b) By considering $\frac{te^{(1-x)t}}{e^t - 1}$, show that $B_n(1 - x) = (-1)^n B_n(x)$ for all $n \geq 0$. ***(5 marks)***

(c) By considering $\frac{te^{(x+1)t}}{e^t - 1} - \frac{te^{xt}}{e^t - 1}$, show that $B_n(x+1) - B_n(x) = nx^{n-1}$ for all $n \geq 0$. Hence show that

$$1^{n-1} + 2^{n-1} + \ldots + N^{n-1} = \frac{B_n(N + 1) - B_n(1)}{n}$$

for all integers $n, N \geq 1$. ***(7 marks)***

(ii) State the Euler-Maclaurin summation formula.

Prove that the sequence

$$\left(\frac{1}{1} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n} \right)_{n=1}^{\infty}$$

converges. ***(10 marks)***

5 (i) Define a character of a finite abelian group. Show that if χ is a non-trivial character of the finite abelian group G, then

$$\sum_{x \in G} \chi(x) = 0.$$

What happens if χ is the trivial character? ***(7 marks)***

(ii) Let n be a positive integer. Explain how the Dirichlet L-function $L(s, \chi)$ for a character χ of $(\mathbb{Z}/n\mathbb{Z})^\times$ is defined, and indicate its region of convergence. ***(3 marks)***

The four mod 10 modular characters $\chi_0, \chi_1, \chi_2, \chi_3$ are listed in the following table:

	χ_0	χ_1	χ_2	χ_3
$n \equiv 1 \pmod{10}$	1	1	1	1
$n \equiv 3 \pmod{10}$	1	i	-1	$-i$
$n \equiv 7 \pmod{10}$	1	$-i$	-1	i
$n \equiv 9 \pmod{10}$	1	-1	1	-1

(a) Show that $0 < \mathrm{Re}(L(1, \chi_1)) < 1$ and $0 < L(1, \chi_2) < 1$. Deduce that $L(1, \chi_j)$ is a non-zero complex number for $j = 1, 2, 3$. ***(7 marks)***

(b) Let p be a prime other than 2 or 5. For each of the possibilities for $p \pmod{10}$, calculate $\chi_0(p) - i\chi_1(p) - \chi_2(p) + i\chi_3(p)$. Assuming

$$\lim_{s \to 1_+} \sum_{\text{primes } p} \frac{\chi(p)}{p^s} = \log L(1, \chi) + \text{ finite constant},$$

deduce that there are infinitely many primes congruent to 3 mod 10. ***(8 marks)***

End of Question Paper