The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Algebraic Topology **2 hours 30 minutes**

*Answer **all five** questions. All questions carry equal weight. You will be marked on correctness/completeness and rigour/presentation. Justify all your answers.*

1 (i) Which of the following maps exhibit covering spaces? Justify your answers briefly.

(a) $p : [0, 1) \longrightarrow S^1 \subset \mathbb{R}^2$ by $t \mapsto (\mathbf{cos} 2\pi t, \mathbf{sin} 2\pi t)$.

(b) $p : S^1 \longrightarrow$ **Möbius Band** by inclusion into the boundary.

(c) The quotient map $S^2 \longrightarrow \mathbb{R}P^2$ which identifies antipodal points.

(ii) Give a cover of $S^1 \vee S^1$ by $S^1 \vee S^1 \vee S^1$, or say why it is not possible.

2 Classify the connected covering spaces of $\mathbb{R}\mathrm{P}^2 \vee \mathbb{R}\mathrm{P}^2$.

3 Let X be the space obtained from two copies of the torus $S_1 \times S_1$ by identifying a circle $S_1 \times \{x_0\}$ in one torus with the corresponding circle $S_1 \times \{x_0\}$ in the other torus.

(i) Compute the fundamental group of X.

(ii) Compute the reduced homology of X and use it to check your answer to part (i). Does this ensure that your answer to part (i) is correct?

4 (i) Explain briefly how a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

(ii) For an exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ of Abelian groups, show that $C = 0$ iff the map $A \xrightarrow{\alpha} B$ is surjective and the map $D \xrightarrow{\delta} E$ is injective. Hence for a subspace A of X, show that the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

(iii) In general, is $H_n(X, A)$ is necessarily isomorphic to $H_n(X/A)$?

5 Recall the Mayer-Vietoris sequence for reduced homology: given spaces $A, B \subset X$ where X is the union of the interiors of A and B, the Mayer-Vietoris exact sequence has the following form:

$$\cdots \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(A \cap B) \longrightarrow \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tilde{H}_0(X) \longrightarrow 0$$

Express S^n as the union of two copies of D^n, and use this exact sequence to compute the homology of S^n by induction. How would you compute the homotopy groups of S^n?

End of Question Paper