The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2010–2011

MAS436: Functional Analysis — 2 hours 30 minutes

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

*Throughout this paper, unless otherwise stated, all normed vector spaces and Hilbert spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$*

**1** *In this question, you may use without proof the fact that a normed vector space where every absolutely convergent series also converges with respect to the norm is a Banach space.*

Let $V$ be a normed vector space.

(i) What is meant by the statement that $V$ is a Banach space ? ***(2 marks)***

(ii) Show that if $V$ is finite-dimensional, then $V$ is a Banach space. ***(6 marks)***

(iii) Show that the space $l^1$ is a Banach space. ***(8 marks)***

(iv) Show that $l^1$ is a linear subspace of $l^2$. ***(3 marks)***

(v) Is $l^1$ a closed subspace of $l^2$ ? Justify your answer. ***(6 marks)***

**2** (i) (a) Let $V$ and $W$ be normed vector spaces. Say what is meant when we describe a linear map $T: V \to W$ as *bounded.* ***(2 marks)***

(b) State the open mapping theorem. ***(3 marks)***

(c) Give an example of a bijective bounded linear map between normed vector spaces which is not open. ***(5 marks)***

(d) State the closed graph theorem. ***(3 marks)***

(e) Deduce the closed graph theorem from the open mapping theorem. ***(6 marks)***

(ii) Let $H$ be a Hilbert space. Let $f, g: H \to H$ be functions where

$$\langle f(v), w\rangle = \langle v, g(w)\rangle$$

for all $v, w \in H$. Prove that $f$ and $g$ are bounded linear maps. ***(6 marks)***

**3** *In this question, you may use without proof the fact that $C_p[0, 2\pi]$ is a dense subset of $L^2[0, 2\pi]$.*

(i) State the Stone-Weierstrass theorem for complex-valued functions.

***(4 marks)***

(ii) Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let $f_k(z) = z^k$. Show that the span of the set $\{z^k \mid k \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$.

***(4 marks)***

(iii) Let $C_p[0, 2\pi]$ be the Banach space of continuous functions $f\colon [0, 2\pi] \to \mathbb{C}$ such that $f(0) = f(2\pi)$, under the norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 2\pi]\}.$$

Write down an isometric isomorphism $\alpha\colon C(\mathbb{T}) \to C_p[0, 2\pi]$. ***(3 marks)***

(iv) Let $e_k(t) = \exp(ikt)$. Show that $\{e_k \mid k \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2[0, 2\pi]$. ***(6 marks)***

(v) Let $f(x) = x$. Find coefficients $a_k \in \mathbb{C}$ such that the series

$$\sum_{k=-\infty}^{\infty} a_k e_k = f$$

in the space $L^2[0, 2\pi]$. ***(4 marks)***

(vi) Use the above to evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

***(4 marks)***

**4** *Throughout this question, you may use without proof the facts that $\|T\| = \|T^*\|$, and that if $\langle x, y\rangle = \|x\|\|y\|$ for vectors $x, y \in H$, then $x$ and $y$ are linearly dependent.*

Let $H$ be a Hilbert space. Let $T\colon H \to H$ be a bounded linear operator such that $\|T\| \leq 1$. Let $T^*\colon H \to H$ be the adjoint operator. Consider the subspaces

$$\begin{aligned} L &= \{x \in H \mid Tx = x\} \\ L_1 &= \{x \in H \mid T^*x = x\} \\ R &= \{x - Tx \mid x \in H\} \end{aligned}$$

(i) Show that $\|T^*x\| = \|x\|$ whenever $x \in L$. ***(4 marks)***

(ii) Show that $L = L_1$. ***(6 marks)***

(iii) Show that $R^\perp = L$. Let $M = \overline{R}$. Deduce that $H = L \oplus M$. ***(6 marks)***

(iv) Let $n \in \mathbb{N}$, and define

$$T_n = \frac{1}{n+1}(I + T + T^2 + \cdots + T^n)$$

Let $x \in R$. Show that $\|T_n x\| \to 0$ as $n \to \infty$. ***(3 marks)***

(v) Let $P\colon H \to H$ be the orthogonal projection onto $L$.

Show that $\|T_n x - Px\| \to 0$ as $n \to \infty$ for all $x \in H$. ***(6 marks)***

**5** *In this question, you can use without proof the fact that the operator $T^*T$ in part (ii) is compact, and relevant theorems from the notes about compact self-adjoint operators.*

(i) Let $H$ be a non-trivial Hilbert space. Let $T\colon H \to H$ be a self-adjoint bounded linear map.

(a) Show that $\|T^*\| = \|T\|$. ***(3 marks)***

(b) Show that the operator $T^*T$ is self-adjoint, and $\|T^*T\| = \|T\|^2$. ***(3 marks)***

(ii) Define $T\colon L^2[0,1] \to L^2[0,1]$ by

$$(Tf)(t) = \int_0^t f(u)\ du.$$

(a) Show that $T^*$ is defined by the formula

$$(T^*g)(s) = \int_s^1 g(u)\ du.$$

***(3 marks)***

(b) Let $f \in C[0,1]$. Show that

$$(T^*Tf)(s) = (1-s)\int_0^s f(u)\ du + \int_s^1 (1-u)f(u)\ du.$$

***(4 marks)***

(c) Let $\lambda$ be a non-zero eigenvalue of $T^*T$. Show that there is an eigenvector $f$, which is twice-differentiable and satisfies

$$\lambda f''(s) = -f(s) \qquad f(1) = 0 \qquad f'(0) = 0.$$

***(4 marks)***

(d) Find all non-zero eigenvalues of $T^*T$. ***(4 marks)***

(e) Calculate $\|T\|$. ***(4 marks)***

**6** *You may use without proof the fact that if $\psi \in L^2(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$ then $\psi * \phi \in L^2(\mathbb{R})$, the convolution formula for the Fourier transform, and the fact that the Fourier transform of a piecewise-continuous integrable function is bounded.*

(i) (a) Let $f, g \in L^1(\mathbb{R})$ be continuous functions. Prove that we have a well-defined integrable function $f * g$ defined by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)\ dt.$$

***(6 marks)***

(b) Show that $(f * g) * h = f * (g * h)$. ***(6 marks)***

(c) Show that $L^1(\mathbb{R})$ is a Banach algebra, with multiplication defined by taking the convolution. ***(4 marks)***

(ii) (a) State the definition of an admissible wavelet. ***(3 marks)***

(b) Let $\psi$ be an admissible wavelet, and let $\phi \in L^1(\mathbb{R})$. Show that the convolution $\psi * \phi$ is an admissible wavelet. ***(6 marks)***

## End of Question Paper