The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2010–2011

Optics and Symplectic Geometry — 2 hours and 30 minutes

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Through the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

1 (i) Define what it means for a $2n \times 2n$ matrix S to be symplectic. ***(2 marks)***

(ii) Prove that the $2n \times 2n$ matrix J is symplectic. ***(3 marks)***

(iii) Calculate the determinant of the $2n \times 2n$ matrix J. ***(7 marks)***

(iv) (a) Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$ be $2n \times 2n$ matrices in block form, where A, B, C, D, A', B', C' and D' denote $n \times n$ matrices. Write down the product

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$$

in block form. ***(3 marks)***

(b) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be $2n \times 2n$ matrix written in block form. Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \quad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \quad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold. ***(10 marks)***

2 (i) Give a definition of a symplectic form on $\mathbb{R}^{2n}$. ***(5 marks)***

(ii) Let ω be a symplectic form on $\mathbb{R}^2$. Show that there is a basis $\{v_1, v_2\}$ for $\mathbb{R}^2$ such that when vectors Z, Z' are expressed as $Z = qv_1 + pv_2$, $Z' = q'v_1 + p'v_2$ then

$$\omega(Z, Z') = qp' - pq'.$$

(10 marks)

(iii) Describe how the coordinates (q, p) of a light ray are introduced. Illustrate this with a sketch. ***(4 marks)***

(iv) Suppose that there are two vertical axis, q_1 and q_2, erected at z_1 and z_2 respectively. Derive the relation between the two coordinates of a ray, (q_1, p_1) and (q_2, p_2) under the assumption that the ray is almost horizontal. Write this relation in matrix form. ***(6 marks)***

3 (i) Consider $\mathbb{R}^{2n}$ with symplectic form ω. Give the definition of a Lagrangian subspace. Define what it means for two Lagrangian subspaces of $\mathbb{R}^{2n}$ to be transversal. Show that two Lagrangian subspaces, L and L', are transversal if and only if $L \cap L' = \{0\}$. ***(5 marks)***

(ii) Verify that the following subspace of $(\mathbb{R}^4, \Omega)$

$$L = \text{span}\{(3, 2, -4, 1), (2, 4, 2, -5)\}$$

is Lagrangian. Is it transversal to $\mathbb{R}^2 \times 0$ and $0 \times \mathbb{R}^2$? ***(9 marks)***

(iii) Denote the standard symplectic form on $\mathbb{R}^{2n}$ by Ω, as usual. On $\mathbb{R}^{4n}$ define $\omega : \mathbb{R}^{4n} \times \mathbb{R}^{4n} \to \mathbb{R}$ by

$$\omega((X_1, X_2), (Y_1, Y_2)) = \Omega(X_1, Y_1) - \Omega(X_2, Y_2),$$

where $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^{2n}$.

(a) Verify that ω is a symplectic form on $\mathbb{R}^{4n}$. ***(5 marks)***

(b) Let S be an invertible $2n \times 2n$ matrix. Show that S is symplectic if and only if the 'graph' of S,

$$G = \{(X, SX) \mid X \in \mathbb{R}^{2n}\}$$

is a Lagrangian subspace of $(\mathbb{R}^{4n}, \omega)$. (You can use without proof that $S \in Sp(2n)$ if and only if $\Omega(X, Y) = \Omega(SX, SY)$ for all $X, Y \in \mathbb{R}^{2n}$. ***(6 marks)***

4 (i) Consider the ray propagation in $\mathbb{R}^3$. In the Cartesian coordinates q_1, q_2, z a light ray crosses the plane $z = z_0$ at point (q_1, q_2), and the plane $z = z_0'$ at point (q_1', q_2'). The angles between the q_1-axis and the ray and between the q_2-axis and the ray are equal to $\frac{\pi}{2} - \varphi_1$ and $\frac{\pi}{2} - \varphi_2$ respectively, where φ_1 and φ_2 are small. Hence, the unit vector v in the ray direction can be approximated by $(\varphi_1, \varphi_2, 1)$. The diffraction index n is constant.

(a) Show that the old, (q_1, q_2, p_1, p_2), and new, (q_1', q_2', p_1', p_2'), coordinates of the ray, where $p_1 = n\varphi_1$ and $p_2 = n\varphi_2$, are related by

$$\begin{bmatrix} q_1' \\ q_2' \\ p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{z_0' - z_0}{n} & 0 \\ 0 & 1 & 0 & \frac{z_0' - z_0}{n} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix}. \qquad (*)$$

(6 marks)

(b) Using the criterion given in part (iv)(b) of question 1 or otherwise prove that the 4×4 matrix in $(*)$ is symplectic. ***(2 marks)***

(ii) The Fermat principle states that the light propagates between two points, A and B, along a path that minimizes the travel time. The speed of light in a medium is c/n, where $c = \text{const}$ is the speed of light in empty space, and $n \geq 1$ is the refraction index of the medium.

(a) You are given that the refraction index is equal to n for $z < 0$, and n' for $z > 0$. Use the Fermat principle to show that the refracted ray is in the plane determined by the incoming ray and the normal to the surface $z = 0$. ***(13 marks)***

(b) The angles between the incoming ray and the z-axis and between the refracted ray and the z-axis are equal to θ and θ'. Use the Fermat principle to show that they are related by the Snell's law:

$$n \sin\theta = n \sin\theta'.$$

(4 marks)

5 The *catenary* is the regular curve $\alpha(t) = (t, \cosh t) = (t, \frac{1}{2}(e^t + e^{-t}))$, $t \in \mathbb{R}$.

(i) Find the wavefront $\bar{\alpha}$ corresponding to an arbitrary constant $C > 0$. Determine the values of $C > 0$ for which the wavefront has singularities and the values of t at which they occur. Simplify the expressions for these values of t as much as possible. (You can use without proof the identity $\cosh^2 t - \sinh^2 t = 1$). ***(14 marks)***

(ii) Find the evolute χ of the catenary and show that it has a singularity at $t = 0$. (You can use without proof the expression for curvature $\kappa(t)$ of curve $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, $\kappa = \dfrac{\alpha_1'\alpha_2'' - \alpha_2'\alpha_1''}{({\alpha_1'}^2 + {\alpha_2'}^2)^{3/2}}$). ***(8 marks)***

(iii) Find an equation for the normal line to the catenary at an arbitrary $t \in \mathbb{R}$ and define a family of curves $F(x, y, t)$ such that for all $t \in \mathbb{R}$, $F(-, -, t)$ is the normal line to the catenary at t. ***(3 marks)***

End of Question Paper