The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-2011**

Mathematics (Numerical Methods and Vector Spaces) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1.

Let $A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -4 & 5 \end{pmatrix}$.

(i) Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. (***6 marks***)

(ii) Verify that L^{-1} and U^{-1} have, respectively, the forms:

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & 1 & 0 \\ \frac{8}{21} & a & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{21} & \frac{4}{65} \\ 0 & \frac{5}{21} & b \\ 0 & 0 & \frac{21}{65} \end{pmatrix}$$

and find the values of a and b. (***4 marks***)

(iii) Explain how you would use the result of part (ii) to find A^{-1}. Given that it has the form:

$$A^{-1} = \frac{1}{65}\begin{pmatrix} 17 & 10 & 4 \\ 10 & c & 10 \\ 8 & 20 & 21 \end{pmatrix}$$

find the value of c. (***2 marks***)

(iv) Derive the formulae for the implicit numerical solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.
(***3 marks***)

(v) If we are given that, in the heat equation of part (iv), $c = 2$ and the boundary and initial conditions are $u(0,t) = 18$, $\frac{\partial u}{\partial t}(3,t) = 0$ for $t \geq 0$, and $u(x,0) = 18 - 12x + 2x^2$ for $0 \leq x \leq 3$, then letting the x-increment be $h = 1$ and the t-increment be $k = 0.5$, set up a table showing the values of u at the grid points for $t = 0$, and hence calculate the values of u at the grid points for $t = 0.5$.
(***10 marks***)

2.

(i) Let $A = \begin{pmatrix} 10 & -2 & 0 & 0 \\ -1 & 10 & -1 & 0 \\ 0 & -1 & 10 & -1 \\ 0 & 0 & -2 & 10 \end{pmatrix}$, $B = \begin{pmatrix} 970 & 196 & 20 & 2 \\ 98 & 980 & 100 & 10 \\ 10 & 100 & 980 & 98 \\ 2 & 20 & 196 & 970 \end{pmatrix}$

Evaluate AB and hence or otherwise find A^{-1}. **(*5 marks*)**

(ii) State whether the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} = 4$$

is elliptic, hyperbolic or parabolic, justifying your answer with an appropriate calculation. **(*2 marks*)**

(iii) The solution of the equation in part (ii) is to be approximated in the rectangular region

$$\left\{(x, y) : 0 \le x \le \frac{3}{2},\ 0 \le y \le 1\right\},$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = -1 \text{ when } x = 0,\ \frac{\partial u}{\partial x} = 2 \text{ when } x = \frac{3}{2},$$
$$u = x^2 - x + y \text{ when } y = 0 \text{ or } 1.$$

Taking $h = k = \frac{1}{2}$, draw a suitable grid for a numerical analysis of the problem, and mark on it the known values of u. Also indicate on your diagram an appropriate notation for the unknown values and any fictitious values you will require to use. **(*6 marks*)**

(iv) Write down equations relating the variables specified in part (iii) and, by eliminating any fictitious values, show that they are of the form $A\mathbf{u} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence, solve the equations for the unknown values of $\mathbf{u}$. **(*12 marks*)**

3.

(i) Give a brief description of a cubic spline, including a clear statement of the conditions that must be satisfied by the two formulae on either side of a datum point. (***6 marks***)

(ii) By deriving appropriate formulae from the requirements detailed in part (i), find the cubic spline which fits the following set of data:

x	0	1	2	3
$f(x)$	10	14	26	36

subject to the additional requirements that the **first** derivative at $x = 0$ is zero and the **second** derivative at $x = 3$ is zero.

(***19 marks***)

You may use without proof that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \quad \text{is} \quad \frac{1}{26}\begin{pmatrix} 15 & -4 & 1 \\ -4 & 8 & -2 \\ 1 & -2 & 7 \end{pmatrix}.$$

4.

(a) Check that the vectors $(1,1,2)$, $(2,-1,0)$ and $(1,0,2)$ form a basis for $\mathbb{R}^3$ but are not an orthogonal set.
Find coordinates c_1, c_2 and c_3 of the vector $f=(1,1,1)$ in this basis. **(*9 marks*)**

(b) Show that if $\{f_i\}_{i=1}^N$ is an orthonormal basis of the complex Hilbert space $\mathbf{V}$, then the norm of the vector

$$f=\sum_{i=1}^N c_i f_i$$

is

$$\|f\|=\left(\sum_{i=1}^N |c_i|^2\right)^{1/2}.$$

Give an expression for the inner product of two vectors

$$f=\sum_{i=1}^N c_i f_i$$

and

$$g=\sum_{i=1}^N d_i f_i$$

in $\mathbf{V}$ in terms of their components with respect to the orthonormal basis

$$\{f_i\}_{i=1}^N$$

(*6 marks*)

(c) Show that the vectors $(\mathrm{j},0,1)$, $(0,1,0)$ and $(-\mathrm{j},0,1)$ form an orthogonal basis of $\mathbb{C}^3$ and form an orthonormal basis using these vectors. Find the coordinates of the vector $g=(\mathrm{j}, 2\mathrm{j}+1, 2)$ in the orthonormal basis. Find the norm of the vector g using the formula of part (b) and check your answer. **(*10 marks*)**

5.

(a) If $\mathbf{V}$ is a Hilbert space with orthonormal basis $\{f_i\}_{i=1}^{N}$, show that the minimum mean square approximation to the vector f in $\mathbf{V}$, using the subset $\{f_i\}_{i=1}^{M}$ where $M < N$, is given by

$$f_M = \sum_{i=1}^{M} c_i f_i$$

where $\{c_i\}$ $(1 \le i \le M)$ are the first M coordinates of f, i.e. $c_i = (f, f_i)$. (***7 marks***)

(b) The Hilbert space of $L^2[0,1]$ functions has an inner product

$$(f,g) = \int f(t) g^*(t)\,\mathrm{d}t$$

where f and g are $L^2[0,1]$ functions and $*$ denotes complex conjugate. Find the constants a, b and c where $a, c > 0$, so that the functions

$$y_1(t) = a \qquad y_2(t) = b + ct$$

form an orthonormal set. (***9 marks***)

(c) The function $f(t) = \mathrm{e}^t$ is to be approximated by an expression of the form

$$f_2 = a\,y_1(t) + b\,y_2(t)$$

with error $e(t)$, where

$$e(t) = f(t) - f_2(t),$$

and $y_1(t)$ and $y_2(t)$ are defined in part (b) above. Find the values of a and b to minimise $\|e\|^2$ and determine $\|e\|^2$. (***9 marks***)

6.

(a) Digital signals of length 4 $(f[0],\ f[1],\ f[2],\ f[3])$ are obtained by sampling a random signal $f(t)$ at times $t = 0, T, 2T$ and $3T$ seconds, where $T = 1/4$.
If the autocorrelation function, $R_f(t)$, of $f(t)$ is given by

$$R_f(t) = \begin{cases} \cos^2 pt & \text{if } |t| \le ½ \\ 0 & \text{otherwise} \end{cases}$$

find the correlation matrix R and show that it is symmetric.
Show that the eigenvalue equation can be written as

$$(4u^2 + 2u - 1)(4u^2 - 2u - 1) = 0$$

where $u = 1 - l$ and l is an eigenvalue. Find the eigenvalues. (***14 marks***)

(b) These digital signals are to be compressed using only one member of the Karhunen-Loeve basis. Find this basis vector and determine the minimum square error associated with this compression. (***11 marks***)

End of Question Paper