The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2011–2012

# Vectors and Mechanics

2 Hours

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** Two points $P$ and $Q$ have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, where

$$\boldsymbol{p} = -\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}, \qquad \boldsymbol{q} = \boldsymbol{i} - 7\boldsymbol{j} + 4\boldsymbol{k}.$$

Find

(i) A unit vector parallel to $\boldsymbol{p} + \boldsymbol{q}$;

(ii) The position vector of the mid-point of the line $PQ$.

***(6 marks)***

**2** Vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are given by

$$\boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}, \qquad \boldsymbol{w} = 3\boldsymbol{i} + \boldsymbol{k}.$$

(i) Find $\boldsymbol{u} \times \boldsymbol{v}$.

(ii) Show that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are coplanar.

***(6 marks)***

**3** Two lines have vector equations

$$\boldsymbol{r}_1 = (1, -1, z) + \lambda\,(-1, 12, 9)\,, \qquad \boldsymbol{r}_2 = (2, -5, -6) + \mu\,(1, -8, -6)\,.$$

Show that the lines only intersect if $z = -3$.
If $z = -3$, find the position vector of their point of intersection. ***(9 marks)***

**4** At time $t$, the position vector $\boldsymbol{r}$ of a particle relative to an origin $O$ is given by $\boldsymbol{r} = x(t)\boldsymbol{i}$, where

$$x(t) = ae^{-ct/2}\left[\cos(\Omega t) + \frac{c}{2\Omega}\sin(\Omega t)\right],$$

where $a$, $c$ and $\Omega$ are positive constants.

(i) What are the dimensions of the constants $a$, $c$ and $\Omega$?

(ii) Find the velocity of the particle at time $t$.

(iii) Show that the particle is at rest at time $t = 0$.

(iv) Find the displacement when the particle is next at rest.

(v) Show that the difference between successive times when the particle is at rest is constant.

***(11 marks)***

**5** A satellite $S$ is situated between the Earth and the Moon, on the straight line joining the Earth and the Moon, a distance $D$ from the Moon.

You are given that the mass of the Earth is approximately 81 times the mass of the Moon.

If the gravitational forces exerted by the Earth and the Moon on the satellite are equal in magitude, show that

$$D = \frac{R}{10},$$

where $R$ is the distance between the Earth and the Moon. ***(7 marks)***

**6** An elastic spring of stiffness 1000 N $\text{m}^{-1}$ is extended 0.05 m.

What is the work done in extending it a *further* 0.05 m? ***(5 marks)***

**7** A stone is dropped from the top of a tower and air resistance can be ignored. The acceleration due to gravity is $g = 9.8$ m $\text{s}^{-2}$.

After one second another stone is thrown vertically downwards from the same point at a speed of 15 m $\text{s}^{-1}$.

If the stones reach the ground simultaneously, find, correct to three significant figures, the height of the tower. ***(5 marks)***

**8** A car of mass 1500 kg moves along a straight horizontal road against a resistance of 500 N. The car has a maximum speed of 180 km $\text{hr}^{-1}$.

At this maximum speed, what is the momentum of the car?

At this maximum speed, what is the kinetic energy of the car?

What is the maximum power of the engine of the car? ***(12 marks)***

**9** A particle of mass $M$ is attached to one end of a light inextensible string of length $2L$.

The other end of the string is attached to a fixed point $O$ a height $L$ above the centre $C$ of a smooth horizontal surface.

The particle moves in a horizontal circle on the surface. The circle has centre $C$. The particle has constant angular speed $\omega$.

(i) Draw a clear diagram showing all the forces acting on the particle.

(ii) Determine the tension in the string in terms of $m$, $L$ and $\omega$.

(iii) Show that the particle remains on the surface provided

$$\omega < \left(\frac{g}{L}\right)^{\frac{1}{2}}.$$

***(13 marks)***

**10** A train of mass $m$ moves along a horizontal straight track against a resistance whose magnitude is $m\,(k + cv)$, where $v$ is the speed of the train, and $c$ and $k$ are positive constants.

If the engine is turned off when the train is moving with speed $U$, show that it will come to rest in a distance $D$ given by

$$D = \frac{1}{c}\left[U - \frac{k}{c}\ln\left(1 + \frac{cU}{k}\right)\right].$$

***(12 marks)***

**11** A tetrahedron $ABCD$ is such that the perpendiculars from the vertices to the opposite faces are concurrent, that is, they intersect in a single point.

Take the origin $O$ to be the point of intersection of the four perpendiculars from the vertices to the opposite faces.

The vertices $A$, $B$, $C$ and $D$ have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively.

(i) Explain why $\boldsymbol{a}.\,(\boldsymbol{b}-\boldsymbol{c})=0$.

(ii) Hence show that

$$\boldsymbol{a}.\boldsymbol{b}=\boldsymbol{a}.\boldsymbol{c}=\boldsymbol{a}.\boldsymbol{d}=\boldsymbol{b}.\boldsymbol{c}=\boldsymbol{b}.\boldsymbol{d}=\boldsymbol{c}.\boldsymbol{d}.$$

(iii) Show that each edge of the tetrahedron is perpendicular to the opposite edge.

(iv) Explain why there is a scalar $\lambda$ such that

$$\boldsymbol{a}-\boldsymbol{d}=\lambda\boldsymbol{b}\times\boldsymbol{c}.$$

*(14 marks)*

## End of Question Paper