The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2011–2012**

**Numbers and Groups** **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*
*You should justify your answers carefully unless the question states otherwise.*

**1** Explain how we can use the Chinese Remainder Theorem to help us solve the following pair of simultaneous congruences. Hence solve it.

$$x \equiv 9 \pmod{17}$$
$$x \equiv 12 \pmod{15}$$

***(10 marks)***

**2** Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions.

(i) Prove that if $f$ and $g$ are injective then the composite $gf$ is injective. ***(5 marks)***

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an injective function. For each $n \in \mathbb{N}$ write $f^n$ for the composite function

$$f \circ f \circ \cdots \circ f : \mathbb{R} \longrightarrow \mathbb{R}$$

where $f$ is applied $n$ times. Prove by induction that $f^n$ is injective for all $n \in \mathbb{N}$. ***(5 marks)***

**3** (i) Let $a_n$ be a sequence of real numbers. State the formal definition of what it means for $a_n$ to converge to $a \in \mathbb{R}$. ***(2 marks)***

(ii) Prove from first principles that the sequence $a_n = 1 + \dfrac{1}{10n}$ converges. You should first say what it converges to. ***(5 marks)***

(iii) Give an example of a sequence that does not converge. Justify your answer briefly. ***(3 marks)***

**4** (i) (a) Let $a \in \mathbb{Z}$. Show that $a^2 \equiv 0 \pmod 9$ if and only if $a$ is divisible by 3. ***(3 marks)***

(b) Consider the relation on $\mathbb{Z}_9$ given by

$$\overline{a} \sim \overline{b} \iff (\overline{a} - \overline{b})^2 = \overline{0}.$$

Is $\sim$ reflexive? symmetric? transitive? an equivalence relation? Justify your answers. ***(7 marks)***

(ii) For $n \in \mathbb{N}$, consider the permutation in $S_{2n}$ given by

$$\beta_{2n} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n \\ n & n-1 & \dots & 2 & 1 & 2n & 2n-1 & \dots & n+1 \end{pmatrix}.$$

For example, $\beta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ and $\beta_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$.

(a) Find the cycle decompositions of $\beta_4$, $\beta_6$ and $\beta_8$.

(b) Show that $\beta_{2n}$ is an even permutation for all $n \in \mathbb{N}$.

***(9 marks)***

(iii) For each of the subsets of $\mathbb{R} \setminus \{0\}$ below, determine whether or not it is a subgroup of $\mathbb{R} \setminus \{0\}$ under multiplication, justifying your answers.

(a) $H_1 = \{x \in \mathbb{R} : x > 0\}$;

(b) $H_2 = \{x \in \mathbb{R} : 0 < x \leq 1\}$;

(c) $H_3 = \mathbb{Z} \setminus \{0\}$.

***(6 marks)***

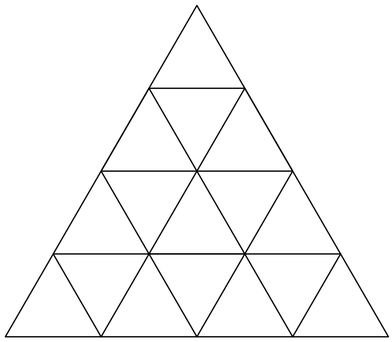
**5** (i) As usual, let $GL_2(\mathbb{Z}_2)$ be the general linear group consisting of all invertible $2 \times 2$ matrices with entries in $\mathbb{Z}_2$. (Recall that any $2 \times 2$ matrix over $\mathbb{Z}_2$ is invertible if and only if its determinant is non-zero.)

Let $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}$.

(a) Show that $A \in GL_2(\mathbb{Z}_2)$. Given that there are 6 matrices in $GL_2(\mathbb{Z}_2)$, list the remaining 5. ***(4 marks)***

(b) Determine the order of $A$. Hence, or otherwise, determine the number of left-cosets of $\langle A \rangle$ in $GL_2(\mathbb{Z}_2)$, stating the name of any theorem that you use. ***(6 marks)***

(ii) A tile, which can be turned over, is to be made by gluing together 16 pieces of coloured plastic, each in the shape of an equilateral triangle, in the shown pattern.



Find the number of essentially different ways in which this can be done with 1 red piece, 6 green pieces and 9 blue pieces. ***(10 marks)***

(iii) Let $p$ be a prime number. Let $G$ be a group of order $p^2$ and let $H$ be a subgroup of $G$. Show that either $H = G$ or $H$ is cyclic. ***(5 marks)***

## End of Question Paper