Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2011–2012**

MAS140 Mathematics (Chemical)
MAS151 Civil Engineering Mathematics
MAS152 Essential Mathematical Skills And Techniques

3 hours

Attempt ***ALL*** *questions.*

Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.

Section A

A1 Use the binomial theorem to evaluate $\lim_{x\to 0} \frac{\sqrt{9-2x}-3}{x}$.

A2 If $f(x) = 2 - 3x$, find $f^{-1}(2t)$.

A3 If $f(x, y) = xy^2 \cosh(x^2 y)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

A4 Evaluate $\lim_{x\to 0} \frac{x - \tan x}{x - \sin x}$.

A5 The complex numbers z_1 and z_2 satisfy

$$Re(z_1 z_2) = Re(z_1)Re(z_2).$$

What (if anything) can you deduce about z_1 and z_2?

A6 If $\mathbf{a} = (4, 2, -3)$ and $\mathbf{b} = (-2, 5, 3)$, find $\mathbf{a}.\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

A7 Compute the definite integral

$$\int_1^e (\ln x)^2 \, dx.$$

A8 Compute the indefinite integral

$$\int \frac{3(\arctan x)^2 - 1}{x^2 + 1} \, dx.$$

A9 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 6 & 7 \end{bmatrix}.$$

Verify that your matrix is the inverse.

A10 Find all the solutions to the system of linear equations

$$\begin{aligned} x - 3y + z &= 5, \\ 6x + 2y - z &= 1, \\ 7x - y &= 6. \end{aligned}$$

A11 Find the general solution to the differential equation

$$x^2 y^2 = \frac{dy}{dx}.$$

A12 Find the specific solution to the differential equation

$$\frac{dy}{dx} = x^4 + x + \frac{3x^2 y}{x^3 + 1}$$

such that $y = 7$ when $x = 1$.

Section B

B1 Find the value of the constants a and b given that

$$\lim_{x\to\infty}\left[\sqrt{ax^2+bx+3}-2x\right]=7.$$

B2 Find all the stationary points and points of inflexion of the function

$$y=\frac{x}{1+x^2}.$$

Sketch the graph of this function.

B3 Express the complex number

$$z=5-6i$$

in polar form.
Find the modulus and argument of each of the three values of $z^{1/3}$ and plot them on the Argand diagram.
Give all your answers correct to two decimal places and use radians for the angles.

B4 The position vector of a particle at time t is given by

$$\mathbf{r}=(6\sin t^2, 6\cos t^2, (1+4t)^{3/2}).$$

Find the velocity of the particle at time t.
Verify that the speed of the particle varies linearly with time.
Find the acceleration of the particle at time $t=0$.

B5 Find the inverse of the matrix

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

B6 Find the general solution to the set of linear differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= 5x_1 + 2x_2 + 4x_3 \\ \frac{dx_2}{dt} &= -3x_1 + 6x_2 + 2x_3 \\ \frac{dx_3}{dt} &= 3x_1 - 3x_2 + x_3. \end{aligned}$$

B7 Compute the indefinite integral

$$\int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\, dx.$$

B8 Find the solution to the differential equation

$$y'' + 4y' + 4y = e^{-2x} + 4\cos 4x$$

such that $y = y' = -1$ when $x = 0$.

End of Question Paper

MAS140-151-152 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \text{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x+y) + \cos(x-y)$

$2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

$\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\tan x = \dfrac{2\tan(x/2)}{1 - \tan^2(x/2)}$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\text{sech} x = \dfrac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\text{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $|x| < 1$

Series

Sum of an arithmetic series:

$$\frac{\textit{first term} \;+\; \textit{last term}}{2} \times (\textit{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r} x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)$$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\mathrm{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$

$\displaystyle\int \frac{dx}{x} = \ln|x|$

$\int e^x \, dx = e^x$

$\int a^x \, dx = \dfrac{a^x}{\ln a} \quad (a > 0, \; a \neq 1)$

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int \sec^2 x \, dx = \tan x$

$\int \mathrm{cosec}^2 x \, dx = -\cot x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

$\int \mathrm{sech}^2 x \, dx = \tanh x$

$\int \mathrm{cosech}^2 x \, dx = -\coth x$

$\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad (|x| < a)$

$\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$

$\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (|x| > a)$

$\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| \quad \left(= \tanh^{-1} \frac{x}{a} \quad \text{if} \;\; |x| < a\right)$

$\displaystyle\int \mathrm{cosec} x \, dx = \ln \tan \left(\frac{x}{2}\right) \quad \text{or} \quad \ln\left(\mathrm{cosec} x - \cot x\right)$

$\displaystyle\int \sec x \, dx = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) \quad \text{or} \quad \ln\left(\sec x + \tan x\right)$

$\displaystyle\int \mathrm{cosech} x \, dx = \ln \tanh \left(\frac{x}{2}\right)$

Integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

Newton-Leibnitz formula

If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a) = F(x)\Big|_a^b$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha,\, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Area of planar figure

The area of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is

$$S = \int_a^b f(x)\, dx$$

Volume of solid of revolution

The volume of a solid of revolution, obtained by rotation of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ through a complete revolution around the x-axis, is

$$V = \pi \int_a^b f^2(x)\, dx$$