The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2011–2012**

Mathematics (Materials) — **3 hours**

All questions are compulsory. The marks awarded to each question or section of question are shown in italics.

1 Identify the number a such that

$$2\log_e 4 - \frac{1}{2}\log_e 4 - 5\log_e 6 + 7\log_e 3 = \log_e a.$$ ***(3 marks)***

2 Complete the square for the following expression:

$$9x^2 + 3x - 2$$ ***(5 marks)***

3 Solve the following inequality for the variable y:

$$|y+2| \leq 4.$$ ***(2 marks)***

4 Expand (remove brackets) and simplify:

$$\frac{y}{2}(y-1) - \frac{3y}{4}(y+2).$$ ***(3 marks)***

5 (i) Write down, as multiples of π, the two values of θ such that $\sin\theta = \frac{1}{\sqrt{2}}$ and $-\pi < \theta \leq \pi$. ***(2 marks)***

(ii) Write $-(\cos\theta + 3\sin\theta)$ in the form $R\cos(\theta - \alpha)$ by clearly stating the values of R and α. ***(8 marks)***

6 Differentiate, with respect to x, the functions

(i) $x^{\log x}$; ***(4 marks)***

(ii) $\dfrac{x}{\sqrt{1+x^2}}$ ***(4 marks)***

7 Find the definite integral

$$\int_0^1 \frac{(\sqrt{x}\ +\ 3)^2}{\sqrt{x}} dx,$$

giving your answer as a fraction. ***(5 marks)***

8 Consider two fixed points $S = (s_1, s_2)$ and $T = (t_1, t_2)$ and a constant $k \neq 1$.

(i) Write down the condition that a point $P(x, y)$ satisfies (length of PS) $=$ k(length of PT). Deduce that P lies on a circle. ***(4 marks)***

(ii) What happens in the case $k = 1$? ***(1 mark)***

(iii) If $S = (0, 0)$ and $T = (1, 0)$ find the points where the circles meet the x-axis in the cases $k = 2$ and $k = 3$. ***(4 marks)***

9 Find the equation that is satisfied by any points of intersection of the line $y = mx$ and the circle

$$(x - 2)^2 + y^2 = 1.$$

For which values of m does the line meet the circle twice? Find the equations of the two tangents from the origin to this circle. ***(5 marks)***

10 (i) Showing your working clearly, find the coefficient of x^2 in the expansion of $(1 + x)^{27}$. ***(2 marks)***

(ii) Use the binomial theorem to evaluate

$$\lim_{x\to\infty} \left[\sqrt{x^2 - 6x + 3} - x\right].$$ ***(3 marks)***

11 Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a} = (4, 1, 0), \quad \mathbf{b} = (-1, 3, 2), \quad \mathbf{c} = (0, 2, 1).$$

Find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and $\mathbf{b} \times (\mathbf{a} \times \mathbf{c})$. ***(5 marks)***

12 Prove that for $x > 1$

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}.$$ ***(5 marks)***

13 Evaluate

$$\int \frac{4x^2 - 13x + 13}{(x+1)(x^2 - 4x + 5)}\, dx.$$ ***(7 marks)***

14 Show that the Maclaurin series for $\sinh x$ is

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$ ***(2 marks)***

Use the ratio test to show that this series converges for all values of x. ***(4 marks)***

15 Find the real numbers x and y which satisfy

$$\frac{x}{4+3i} - \frac{6}{y-i} = 1.$$ ***(7 marks)***

16 The matrix A is defined by

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Show that the equation $AX = \lambda X$, where λ is a constant and

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

can be written as

$$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 4 & -1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ ***(2 marks)***

Find all three values of λ for which non-zero solutions X can exist. ***(5 marks)***

For each of these values of λ, find the corresponding solution X. ***(8 marks)***

End of Question Paper

Formula Sheet for MAS153/MAS157 Examination

These results may be quoted without proof, unless proofs are asked for in the question.

Trigonometry

For any angles A and B

$\sin^2 A + \cos^2 A = 1$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Coordinate Geometry

The acute angle α between lines with gradients m_1 and m_2 satisfies

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \qquad (m_1 m_2 \neq -1)$$

while the lines are perpendicular if $m_1 m_2 = -1$.
The equation of a circle centre (x_0, y_0) and radius a is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

Hyperbolic Functions

$\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x + \tanh^2 x = 1$

$\cosh^2 x + \sinh^2 x = \cosh 2x$

$2 \sinh x \cosh x = \sinh 2x$

$\cosh^2 x = (1 + \cosh 2x)/2$

$\sinh^2 x = -(1 - \cosh 2x)/2$

Differentiation

Function (y)	**Derivative** (dy/dx)
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

NB. It is assumed that x takes only those values for which the functions are defined.

For u and v functions of x, and with $u' = \dfrac{du}{dx}$, $v' = \dfrac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}.$$

Integration

In the following table the constants of integration have been omitted.

Function $f(x)$	**Integral** $\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$n \neq -1$
ae^{ax}	e^{ax}	
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
$a\sin ax$	$-\cos ax$	
$a\cos ax$	$\sin ax$	
$a\tan ax$	$\ln\lvert\sec ax\rvert$	
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert$	

Integration by parts

$$\int uV\,dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx}\,dx$$

or

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Series

Binomial Theorem: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \binom{n}{r}x^r + \cdots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

If n is a positive integer, the series terminates and is convergent for all x.

If n is not a positive integer, the series is infinite and converges for $|x| < 1$.

Taylor expansion of $f(x)$ about $x = a$ is

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Maclaurin expansion of $f(x)$ is

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Alternating Series Test

The series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $a_1, a_2, a_3, a_4, \ldots$ are all positive, converges if $a_1 > a_2 > a_3 > \cdots$ and $a_n \to 0$ as $n \to \infty$.

Ratio Test

If the series $a_1 + a_2 + a_3 + a_4 + \cdots$ satisfies

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda,$$

then

1. if $\lambda > 1$, the series diverges,
2. if $\lambda < 1$, the series converges.

Vectors

If vectors $\mathbf{a}$ and $\mathbf{b}$ are given in Cartesian component form by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

the scalar product $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and the vector product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

If a plane passes through a point with position vector $\mathbf{a}$, and is normal to the vector $\mathbf{n}$, then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

where $\mathbf{r} = (x, y, z)$.