Data provided: Formula Sheet

MAS156

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

MAS156 Mathematics (Electrical) **3 hours**

*Attempt **ALL** questions.*
Each question in Section A carries 3 marks and each question in Section B carries 8 marks.

Section A

A1 Determine whether the function $f(x) = [\sin(\pi + x) - \sin(\pi - x)]^2$ is even, odd or neither.

A2 State the fundamental period of each of the following functions.

(i) $f(x) = \cos(2x + \pi)$;

(ii) $g(x) = \tan\left(\frac{x}{3}\right)$;

(iii) $h(x) = \sin(\omega x + \phi)$.

A3 Find the modulus and principal argument of $\dfrac{1+j}{2j-1}$.

A4 Find the first and second derivatives of $\ln(2 + \cos x)$.

A5 Let $y = \cosh x$. Show that $y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

A6 Let $\mathbf{a} = (2, 0, 1)$, $\mathbf{b} = (-1, -1, -1)$ and $\mathbf{c} = (1, 2, 0)$. Calculate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

A7 Evaluate $\displaystyle\int \frac{\sin x \cos x}{(1 + \sin^2 x)^2}\, dx$.

A8 Evaluate $\displaystyle\int \ln(x^2 + 3)\, dx$.

A9 Solve the differential equation

$$(x + 2) \tan y \frac{dy}{dx} = 1,$$

given that $y(0) = \pi/4$.

A10 Find the inverse Laplace transform of the function $\dfrac{2s + 3}{s^2 + 2s + 10}$.

A11 Find $\displaystyle\lim_{x \to 0} \frac{\ln(1 + 2x^2)}{\sin^2 x}$.

A12 If $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ evaluate any of $AB,\ A^T B,\ AB^T,\ A^T B^T$ which exist.

Section B

B1 By factorizing $a^2 + a - 2$, or otherwise, find all six solutions to $z^6 + z^3 - 2 = 0$ in exponential form and show that the product of these six solutions is -2. Can you find another justification for this result?

B2 Find the amplitude, $A(t)$, and phase, $\theta(t)$, of $z(t) = (\cos t)e^{tj/2}$ for $0 \le t \le 2\pi$, drawing graphs of both quantities. For which values of t is $0 \le A(t) \le \frac{1}{2}$?

B3 Let $z = \sinh(xy)$. Calculate $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2}$.

Now make the substitution $x = u + v$, $y = u - v$ in z, and calculate $\dfrac{\partial^2 z}{\partial u \partial v}$. What do you notice?

B4 The position vector of a particle at time t is given by $\mathbf{r} = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}$. Show that the acceleration $\mathbf{a}$ of the particle is perpendicular to the velocity $\mathbf{v}$, and that $\mathbf{a} = -\omega^2\mathbf{r}$. What is the magnitude of the acceleration if $\omega = \pi$?

B5 Evaluate $\displaystyle\int \frac{x^2 + 6x + 20}{(x+3)(2x^2 + 8x + 17)}\, dx$.

B6 Find the solution to the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = x^2$$

when $y(0) = 0$ and $y'(0) = 0$.

B7 Devise a Newton-Raphson iterative scheme to solve the equation

$$x \ln x = 7.$$

Find x_0, the integer which is nearest the root. Use your scheme to find the root correct to three decimal places.

For what range of values for x_0 will your scheme converge to the root?

B8 Find the relationship between α and β if the system of equations

$$\begin{aligned} x + 2y - 3z &= 0 \\ 3x + y + z &= 0 \\ 2x + \alpha y + \beta z &= 0 \end{aligned}$$

has a non-trivial solution.

Find the general solution when $\alpha = 2$ and $\beta = -2$.

Find α and β if the equations

$$\begin{aligned} x + 2y - 3z &= 0 \\ 3x + y + z &= 1 \\ 2x + \alpha y + \beta z &= 0 \end{aligned}$$

have infinitely many solutions.

End of Question Paper

Formula Sheet for MAS156

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \text{cosec}^2\theta$
$\cos^2\theta = (1 + \cos 2\theta)/2$
$\sin^2\theta = (1 - \cos 2\theta)/2$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ where
$R = \sqrt{a^2 + b^2}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Complex Numbers ($j = \sqrt{-1}$)

Euler's Relation: $e^{j\theta} = \cos\theta + j\sin\theta$
De Moivre's Theorem:
$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$

Hyperbolic Functions

$\cosh x = \frac{1}{2}(e^x + e^{-x})$
$\sinh x = \frac{1}{2}(e^x - e^{-x})$
$\cosh^2 x - \sinh^2 x = 1$
$\text{sech}^2 x + \tanh^2 x = 1$
$\cosh^2 x + \sinh^2 x = \cosh 2x$
$2\sinh x \cosh x = \sinh 2x$
$\cosh^2 x = (1 + \cosh 2x)/2$
$\sinh^2 x = -(1 - \cosh 2x)/2$

Series

Sum of an arithmetic series:

$$\frac{\textit{first term} + \textit{last term}}{2} \times (\textit{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\}\text{ valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \le 1)$$

Vectors

For vectors $\mathbf{a} \equiv a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} \equiv a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$:

Magnitude and unit vector:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Scalar product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

Vector Product:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\sin\theta\;\hat{\mathbf{n}} \\ &= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k} \end{aligned}$$

Equation of a line:
The equation of a line in the direction of $\mathbf{a}$, and passing through the point with position vector $\mathbf{r}_0$, is given by

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{a}$$

where λ is a variable parameter.

General Differentiation Formulae

$$\begin{aligned} \frac{d\,(uv)}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{vu' - uv'}{v^2} \\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \end{aligned}$$

Differentiation

Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\mathrm{cosec}^2\, x$
$\sec x$	$\sec x \tan x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2\, x$
$\coth x$	$-\mathrm{cosech}^2\, x$
$\mathrm{sech}\, x$	$-\mathrm{sech}\, x \tanh x$
$\mathrm{cosech}\, x$	$-\mathrm{cosech}\, x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$

Integration

Function	**Integral**
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$
$\operatorname{cosec} x$	$\ln\tan\left(\dfrac{x}{2}\right)$ or $\ln(\operatorname{cosec} x - \cot x)$
$\sec x$	$\ln\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$ or $\ln(\sec x + \tan x)$
$\operatorname{cosech} x$	$\ln\tanh\left(\dfrac{x}{2}\right)$

If $t = \tan\left(\dfrac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

Integration-by-parts

$$\int_a^b uV\,dx = [u \times (integral\ of\ V)]_a^b - \int_a^b (integral\ of\ V) \times \frac{du}{dx}dx$$

or $\displaystyle\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

Taylor expansion of $f(x)$ about $x = a$

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \ldots$$

Maclaurin expansion of $f(x)$

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \ldots$$

Newton-Raphson formula for the root of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}(f(t))$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, 2, \ldots)$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$e^{at}f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$