The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Advanced Calculus - MAS202 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The random variables X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 4e^{-2x-2y} & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise} . \end{cases}$$

Calculate the probability $P(X + Y \leq 1)$. ***(9 marks)***

(ii) Let $\omega = (3x^2 + y^2)\, dx + (4y^3 + 2xy)\, dy$.

(a) Show, *without* finding a potential function, that ω is an exact differential.

(b) Now find a potential function f for ω.

(c) Evaluate the line integral $\int_\gamma \omega$, where

$\gamma : x = \cos^7(2t), y = \sin^3(2t),\ 0 \leq t \leq \frac{\pi}{2}$. ***(10 marks)***

(iii) A function F is defined by the formula

$$F(x) = \int_0^{x+e^x} e^{xt^2}\, dt.$$

Write down an expression for the derivative $\frac{dF}{dx}$. (Do not attempt to evaluate the integral in your expression.) ***(6 marks)***

2 (i) State Green's Theorem, being careful to include any conditions needed for its validity. Suppose that C is a curve and D is a region satisfying Green's Theorem, deduce that the area of D is equal to the line integral $\int_C x\,dy$.
(7 marks)

(ii) Let C be the circle of radius R centered at the origin. *Using the formula you proved in part (i)*, calculate the area enclosed by C. (You should recognize the answer.) ***(9 marks)***

(iii) For C as in part (ii), calculate directly the line integral

$$\int_C P\,dx + Q\,dy\,,$$

where

$$P = \frac{-3y}{x^2+y^2},\ Q = \frac{3x}{x^2+y^2}.$$

Calculate also $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. Why does this not contradict Green's Theorem?
(9 marks)

3 (i) Let f be the periodic function with period 2π such that $f(x) = 2|x|$ for $-\pi \leq x \leq \pi$.

(a) Sketch the graph of f. Calculate all the coefficients in the Fourier series for f. ***(15 marks)***

(b) What can you deduce by plugging in $x = 0$? ***(6 marks)***

(ii) Let $g(x)$ be the periodic function with period 2π such that $g(x) = 4|x|$ for $-\pi \leq x \leq \pi$. Using (i) write down the coefficients of the Fourier series for $g(x)$. ***(4 marks)***

4 (i) If X is a random variable, define the probability generating function $G_X(s)$. Prove that if X and Y are independent then $G_{X+Y}(s) = G_X(s)G_Y(s)$.

Suppose that $P(X = k) = \dfrac{1}{k!e}$ for all integers $k \geq 0$. Show that $G_X(s) = e^{s-1}$. Let Y be another random variable, independent of X, but with the same distribution. Obtain an explicit value for the probability $P(X + Y = k)$. ***(9 marks)***

(ii) Recall the Fourier transform $\hat{f}(s) = \displaystyle\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$ (when it exists), and the Fourier inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{itx}\,dx$$

(valid under certain conditions). Let

$$u(t) = e^{-|t|} = \begin{cases} e^{-t}, & \text{if } t \geq 0; \\ e^{t}, & \text{if } t < 0. \end{cases}$$

(a) Show that $\hat{u}(s) = \dfrac{2}{1+s^2}$. ***(8 marks)***

(b) Evaluate the integral $\displaystyle\int_{0}^{\infty} \frac{\cos(2x)}{1+x^2}\,dx$. [Hint: use Fourier inversion and $e^{i\theta} = \cos\theta + i\sin\theta$.] ***(8 marks)***

End of Question Paper