The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2011–12**

**MECHANICS** **2 hours**

*Attempt all THREE questions.*

**1** (i) (a) Sketch the region, $R$, that is bounded by the curve $y = x^2$ and the line $y = 4$. ***(2 marks)***

(b) Evaluate the double integral of $f(x, y) = 6x^2 + 2y$ over $R$. ***(7 marks)***

(ii) Consider the double integral

$$I = \int_{y=0}^{2} \int_{x=\frac{y}{2}}^{1} e^{x^2} \, dxdy.$$

(a) Sketch the region over which the integration takes place. ***(2 marks)***

(b) By reversing the order of integration, evaluate $I$. ***(7 marks)***

(iii) Find, in regular cartesian form, the equation of the tangent plane to the surface $x = y^3 + z^3$ at the point $(7, 2, -1)$. ***(7 marks)***

**2** (i) A particle of mass $m$ moves in a plane with origin $O$, and its plane polar coordinates are $(r(t), \theta(t))$. It is subject to a force directed towards $O$ of magnitude $mF(r)$. You are **given** that the radial and transverse components of Newton's Second Law reduce to:

$$\ddot{r} - r\dot{\theta}^2 = -F(r) \tag{1}$$

$$r^2\dot{\theta} = h \tag{2}$$

where $h$ is a positive constant (and, in the usual notation, a dot over a variable denotes its time derivative). Make the substitution $u = r^{-1}$, and use (2) to show that

$$\dot{r} = -h\frac{du}{d\theta}$$

and deduce that (1) becomes

$$h^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^{-2}F(u^{-1}). \tag{3}$$

***(11 marks)***

(ii) Calculate $\nabla\phi$, where, in the usual notation, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $r = |r| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, for

(a) $\phi = \ln|r|$,

(b) $\phi = \dfrac{1}{r}$. ***(14 marks)***

**3** A thin uniform rod $AB$ of length $2a$ and mass $M$ rotates freely about an axis $CD$ which is perpendicular to the length of the rod and passes through its centre $O$.

(i) By considering a small element of rod of length $\delta x$ a distance $x$ from $O$, derive the result that the moment of inertia of the rod about the axis $CD$ is $\frac{1}{3}Ma^2$, where $M$ is the mass of the rod. ***(7 marks)***

(ii) A pendulum consists of such a rod pivoted on a horizontal axis through its midpoint together with a small regulating mass $m$ at a distance $x$ from the midpoint.

(a) Show that the period $\tau$ of small oscillations is given by

$$\tau = 2\pi\left(\frac{Ma^2 + 3mx^2}{3mgx}\right)^{\frac{1}{2}}.$$

***(10 marks)***

(b) Prove that if $M > 3m$ the period is always lengthened when $x$ is decreased slightly. (Hint: start by considering $\dfrac{d(\tau^2)}{dx} = 0$.)

***(8 marks)***

## End of Question Paper