Data provided: "Statistics Tables" by H.R. Neave

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Statistics Core **2 hours**

RESTRICTED OPEN BOOK EXAMINATION

Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–25; Q2–19; Q3–28; Q4–15; Q5–13)

1 Let X be a $Be(3,2)$ random variable, with probability density function $f_X(x)$ given by

$$f_X(x) = \begin{cases} 12x^2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the coefficient of skewness of X. ***(7 marks)***

(b) Let $Y = -\log_e X$. Find the probability density function of Y. ***(8 marks)***

(c) Find $P\left(\frac{1}{2} - t \le X \le \frac{1}{2} + t\right)$ if $0 < t < \frac{1}{2}$. ***(4 marks)***

(d) Let $T = \left|X - \frac{1}{2}\right|$. Using your answer to (c),

(i) write down $P(T \le t)$ for $0 < t < \frac{1}{2}$; ***(1 mark)***

(ii) show that the probability density function of T is

$$f_T(t) = \begin{cases} 3 - 12t^2 & 0 < t < 1/2 \\ 0 & t < 0 \text{ or } t > \frac{1}{2} \end{cases}$$

(5 marks)

2 Let $T \subseteq \mathbb{R}^2$ be the set $\{(x,y) : 0 \leq x \leq 1, -\pi \leq y \leq \pi\}$, and let X and Y be random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} k(x+1+\cos y) & (x,y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of k. ***(7 marks)***

(b) Find the marginal probability density functions of X and Y, and find the conditional probability density function of X given $Y = y$. ***(12 marks)***

3 Let the random vector $\mathbf{X} = (X, Y)^T$ have a bivariate normal distribution with the marginal distributions of X and Y both being standard normal and the correlation between X and Y being $1/2$.

(a) What are the mean vector and covariance matrix of $\mathbf{X}$? ***(3 marks)***

(b) Let $W = 4X - Y + 3$ and $Z = kX + Y$, and let $\mathbf{W} = (W, Z)^T$

(i) Find the mean vector and covariance matrix of $\mathbf{W}$, and hence give the distributions of W and Z in the form $N(\mu, \sigma_2)$. ***(10 marks)***

(ii) For what value of k are W and Z independent? ***(4 marks)***

(c) Let $U = \exp(X+Y)$ and $V = \exp(X-Y)$. Find the joint probability density function of the random vector $(U,V)^T$, stating clearly the values for which it is non-zero. ***(11 marks)***

4 Let $x_1, x_2, \ldots, x_n$ be a random sample from a distribution with probability density function

$$f(x) = \begin{cases} \dfrac{\theta}{x^{\theta+1}} & x > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown. (This is an example of a Pareto distribution.)

(a) Find the likelihood of θ given the data $x_1, x_2, \ldots, x_n$. ***(3 marks)***

(b) Find the maximum likelihood estimate of θ given the data $x_1, x_2, \ldots, x_n$. ***(12 marks)***

5 A Negative Binomial distribution with parameters m and θ has probability function

$$p(x) = \binom{m+x-1}{x} \theta^m (1-\theta)^x,$$

for x a non-negative integer, with $0 \leq p \leq 1$ and m a positive integer. An observation from this distribution gives the value 1.

(a) Assuming that θ is known to be $1/3$ and that m is unknown, write down an expression for the likelihood of m given this observation, and calculate its value for $m = 1, 2, 3, 4$. ***(5 marks)***

(b) What is the maximum likelihood estimate of m given this observation? ***(8 marks)***

End of Question Paper