The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2011-2012

Mathematics III(Electrical) — 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 z is the complex number $x + jy$.

(i) Sketch the four regions in the z-plane corresponding to $y \leq -2$, $y \geq x - 2$, $|z| \leq 2$ and $|z - 3| \leq 1$. ***(8 marks)***

(ii) For the mapping $w = (1 - j)z - 1 - j$ where $w = u + jv$,

(a) find $u(x, y)$ and $v(x, y)$;

(b) find the magnification, rotation and translation for this mapping and hence find the image in the w-plane of the region $|z| \leq \dfrac{1}{\sqrt{2}}$ in the z-plane;

(c) using the algebraic or geometric method, find the image in the w-plane of the line $y = x$ in the z-plane;

(d) sketch your results for (b) and (c) in the z- and w-planes;

(e) find the fixed points of the mapping. ***(17 marks)***

2 Expand the function $f(z) = \dfrac{j}{z(z + j)}$ into partial fractions and hence

(i) find the first three non-zero terms of the power series expansion of $f(z)$ about the point $z = j$ using the Taylor series and the binomial expansion method. Show the region of convergence of the power series and all poles and zeros of $f(z)$ on the Argand diagram. ***(18 marks)***

(ii) Find the first four terms of the Laurent series expansion of $f(z)$ about the point $z = -j$. ***(7 marks)***

3 (i) Find all the poles of $f(z) = \dfrac{1}{z(z+j)(z+1-j)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint\limits_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, where

(a) C is the circle $|z-3| = 1$

(b) C is the circle $|z| = 1.3$.

(16 marks)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx.$$

(9 marks)

4 (i) The function $y(t)$ satisfies the differential equation

$$\ddot{y} - 2\dot{y} + y = 3\delta(t-2) + e^{3t}$$

(where dot denotes differentiation with respect to t) and the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$. Show that the Laplace transform, $Y(s)$, of $y(t)$ is given by

$$Y(s) = \frac{1 + 3e^{-2s} + \frac{1}{s-3}}{s^2 - 2s + 1}$$

Hence determine $y(t)$ for $t > 0$. ***(13 marks)***

(ii) Sketch the function $f(t) = f_1(t) + f_2(t)$ where $f_1(t) = e^{2t}H(-t)$ and $f_2(t) = e^{-t}H(t)$ where $H(t)$ is the step function. Using direct integration, show that the Fourier transform of $f(t)$ is $\dfrac{3}{(2-j\omega)(1+j\omega)}$.

A second function, $g(t)$, is given by $g(t) = 2f_1(t) - f_2(t)$. Write down its Fourier transform. Write $g(t)$ in terms of $f(t)$, and hence verify the differentiation property of the Fourier Transform. (Do not worry about the lack of differentiability at $t = 0$.) ***(12 marks)***

End of Question Paper

MAS242 FORMULA SHEET

Table of Laplace Transforms

$f(t)$	$F(s)$	Region of validity
constant $= c$	$\frac{c}{s}$	$Re(s) > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$Re(s) > \alpha$
t	$\frac{1}{s^2}$	$Re(s) > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
$e^{\alpha t} \sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$e^{\alpha t} \cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$\delta(t-T)$	e^{-sT}	delta function
$H(t-T)$	$\frac{e^{-sT}}{s}$	step function
$H(t) - H(t-T)$	$\frac{1}{s}(1 - e^{-sT})$	rectangular pulse

Note: in this table the parameters α and k are real constants and H is the Heaviside step function.

Some general properties of the Laplace transform

In the following table the notation $\mathbf{L}\{f(t)\} = F(s)$ has been used.

$\mathbf{L}\{af(t) + bg(t)\} = a\mathbf{L}\{f(t)\} + b\mathbf{L}\{g(t)\}$	linearity
$\mathbf{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathbf{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0) - f'(0)$	differentiation twice with respect to t
If $g(t) = \int_0^t f(u)du$ then $\mathbf{L}\{g(t)\} = \frac{1}{s}F(s)$	integration
$\mathbf{L}\{tf(t)\} = -\frac{dF}{ds}$	differentiation w.r.t. s
$\mathbf{L}\{e^{-kt}f(t)\} = F(k+s)$	shift
$\mathbf{L}\{f(at)\} = \frac{1}{\|a\|}F\left(\frac{s}{a}\right)$	scaling
$\mathbf{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$	time delay

Convolution: For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Residues

The formula for the residue at a pole, z_0, of order 1 is

$$\lim_{z \to z_0}[(z - z_0)f(z)]$$

The general formula for the residue at a pole, z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z \to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right).$$

Table of Fourier Transforms

Time Domain		**Frequency Domain**
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$		$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

Standard Functions

$e^{-\alpha t}$ for $\alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
e^{-at^2} for $a > 0$	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$\Pi(t) = \begin{cases} 1 & \text{for } \lvert t\rvert<\frac{1}{2} \\ 0 & \text{for } \lvert t\rvert>\frac{1}{2} \end{cases}$	$sinc\frac{\omega}{2} = \begin{cases} \frac{sin(\omega/2)}{\omega/2} & \text{for } \omega\neq 0 \\ 1 & \text{for } \omega=0 \end{cases}$
$\triangle(t) = \begin{cases} 1-\lvert t\rvert & \text{for } \lvert t\rvert<1 \\ 0 & \text{for } \lvert t\rvert>1 \end{cases}$	$sinc^2\frac{\omega}{2}$

Some general results

$\left.\begin{matrix} f(t) \\ F(t) \end{matrix}\right\}$	symmetry	$\begin{cases} F(\omega) \\ 2\pi f(-\omega) \end{cases}$
$\frac{df}{dt}$	differentiation	$j\omega F(\omega)$
$f(t-\tau)$	time shift	$e^{-j\omega\tau}F(\omega)$
$e^{j\theta t}f(t)$	frequency shift	$F(\omega-\theta)$
$f(at)$	scaling	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
$\int_{-\infty}^{\infty} f(u)g(t-u)du$	convolution	$F(\omega)G(\omega)$

Generalised functions

1		$2\pi\delta(\omega)$
$\delta(t)$		1
$\delta(t-\tau)$		$e^{-j\omega t}$
$e^{j\theta t}$		$2\pi\delta(\omega-\theta)$
$\text{III}\left(\frac{t}{\tau}\right)$		$\tau\text{III}\left(\frac{\omega\tau}{2\pi}\right)$
for the 'Shah' function	$\text{III}(t) = \sum_{n=-\infty}^{\infty}\delta(t-n)$	