Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find and classify the stationary point of the function

$$f(x, y) = x^2 + y^2 + 8x + 8.$$

(9 marks)

(ii) Write down the iteration formula for the Newton-Raphson method. Starting from $x_0 = 1$ perform four iterations of the Newton-Raphson method to find an approximation to a root of the equation

$$f(x) = x^3 + x - 1 = 0.$$

Work correct to six decimal places throughout. *(9 marks)*

(iii) State one advantage and one disadvantage of the bisection method, compared to the Newton-Raphson method, for finding a root of a function. The function $g(x) = 3x - e^{-x}$ has a root between $x = 0.25$ and $x = 0.27$. Use the bisection method to calculate the value of this root correct to three decimal places. *(7 marks)*

2 A periodic function $f(x)$ of period 2π is defined by

$$f\,(x) = \begin{cases} -k \text{ for } & -\pi \le x < 0 \\ k \text{ for } & 0 \le x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x),$$

where k is a constant.

Show that the first four non-zero terms of the Fourier series expansion of $f(x)$ are given by

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x...\right).$$

(14 marks)

Sketch a graph of $f(x)$ for $-3\pi \le x \le 3\pi$. ***(6 marks)***

Use the Fourier series expansion of $f(x)$ to deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

(5 marks)

3 (i) Find the directional derivative of the function

$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$

in the direction of the vector $\mathbf{v} = (1, 0, -2)$ at the point $(2, 1, 3)$.

(9 marks)

(ii) For the vector

$$\mathbf{a} = (ye^x, e^x, z^2),$$

find a scalar potential ψ such that

$$\mathbf{a} = \nabla\psi.$$

(11 marks)

(iii) Find the curl of the vector $\mathbf{w} = (yz, 3xz, z)$. ***(5 marks)***

4 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} + 5\frac{\partial^2 y}{\partial x \partial t} - 6\frac{\partial^2 y}{\partial x^2} = 0,$$

has solutions of the form $y = f(x + \lambda t)$ for arbitrary functions f provided that $\lambda = -6$ or $\lambda = 1$. ***(8 marks)***

Give an interpretation, including a clear diagram, of the form of the solution in each case. ***(6 marks)***

Derive the solution that satisfies the conditions

$$y(x, 0) = 0$$

$$\frac{\partial y}{\partial t}(x, 0) = 5x.$$

(11 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \le x \le L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \le x \le L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if $f_x = f_y = 0$ at (x_0, y_0).

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if
$f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at (x_0, y_0),

(ii) a maximum if
$f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ at (x_0, y_0),

(iii) a saddle point if
$f_{xx}f_{yy} - f_{xy}^2 < 0$ at (x_0, y_0).