Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Mathematics II (Materials) **2 hours**

Marks will be awarded for answers to all questions in Section A, and for your best ***THREE*** *answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.*

Section A

A1 Find the solution of the equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}$$

for $x > 0$ which satisfies $y = 1$ when $x = 1$. ***(8 marks)***

A2 Find the general solution of the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 4e^{2x}$$

(6 marks)

A3 A rectangular sided box has sides of length $x = 2\,\mathrm{m}$, $y = 3\,\mathrm{m}$ and $z = 1\,\mathrm{m}$. If the lengths of the sides are changed to $2.1\,\mathrm{m}$, $3.1\,\mathrm{m}$ and $1.1\,\mathrm{m}$ respectively, use the chain rule to find the approximate percentage increase in the volume of the box. ***(8 marks)***

A4 The following table shows the MAS154 marks ($= y$) and the A level points scores ($= x$) of 10 students:

x	350	460	290	380	380	330	280	370	280	260
y	64	68	30	67	54	26	31	57	57	51

Showing your working, calculate the mean and standard deviation of x and y, and also the correlation between x and y. Comment briefly on the implications of the correlation between x and y. ***(11 marks)***

A5 A scalar field ϕ is given by

$$\phi = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}.$$

Find $\nabla\phi$, and hence find a unit vector normal to the surface $\phi = 3$ at the point $(2, -3, 4)$.

Find also the directional derivative of ϕ in the direction normal to this surface. ***(7 marks)***

Section B

B1 (a) Use the substitution $y = xz$ to find, for $x > 0$, the solution y of the equation

$$x\frac{dy}{dx} = y + \frac{(xy)^{1/2}}{x+1}$$

which satisfies $y = 1$ when $x = 1$. ***(14 marks)***

(b) Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = x.$$ ***(6 marks)***

B2 (a) Let $R = \left(x^2 + y^2\right)^{1/2}$, and define a scalar field ϕ by

$$\phi = \ln R.$$

Find $\nabla\phi$, and show that

$$\nabla^2\phi = 0.$$ ***(14 marks)***

(b) Measurements of two quantities x and y are made, giving means of 3.7 and 13.9 respectively, variances of 1.3 and 6.7 respectively, and a covariance between x and y of -2.9.

It is assumed that x and y satisfy the linear relationship

$$y = a + b(x - \overline{x}), \qquad (*)$$

where $\overline{x}$ is the mean of x.

Calculate the least squares estimates of a and b, correct to 3 significant figures. State, giving reasons, whether you expect $(*)$ to give a good model. ***(6 marks)***

B3 The function $f(x) = x^2$ is defined on the interval $0 \leqslant x \leqslant 1$.

(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi x}{n} - \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{\sin(2m-1)\pi x}{(2m-1)^3}.$$ ***(17 marks)***

(b) Sketch the function given by the above Fourier series on the interval $-3 \leqslant x \leqslant 3$. ***(3 marks)***

B4 (a) Show that the function

$$\phi(x,t) = f(x - ct) + g(x + ct),$$

where f and g can be any functions, and c is a constant, satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}.$$ ***(5 marks)***

(b) Using the result of part (a), find the solution to the wave equation which satisfies

$$\phi(x,0) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial t}(x,0) = akc\cos kx,$$

where k and a are constants. ***(11 marks)***

If $\phi(x,t)$ represents the transverse displacement at time t of an infinitely long string, describe the motion of the string. ***(4 marks)***

End of Question Paper

FORMULA SHEET

Trigonometry

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$, where $R = \sqrt{(a^2 + b^2)}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Hyperbolic Functions

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$

$\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x + \tanh^2 x = 1$

$2\sinh x\cosh x = \sinh 2x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\sinh^{-1} x = \ln\left[x + \sqrt{(1 + x^2)}\right], \quad \text{all } x$

$\cosh^{-1} x = \ln\left[x + \sqrt{(x^2 - 1)}\right], \quad x \geqslant 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right), \quad |x| < 1$

$\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x + 1}{x - 1}\right), \quad |x| > 1$

Differentiation and Integration

Function	**Derivative**
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec} x$	$-\text{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech} x \tanh x$
$\text{cosech} x$	$-\text{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad \mid x \mid< 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \quad \mid x \mid> 1$

Function	**Integral**
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uv dx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v) dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1x} + Be^{m_2x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal ($m_1 = m_2 = m$),

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex ($m_1 = p + iq, m_2 = p - iq$), where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\qquad$ $y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\qquad$ $y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\qquad$ $y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\qquad$ $y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\qquad$ $y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} \;+\; b_n \sin\frac{n\pi x}{l}\right) \;,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0,1,2,\ldots\,,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0,1,2,\ldots\,.$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;,\quad a_n \;=\; \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the Fourier sine series is

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;,\quad b_n \;=\; \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi \;=\; \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \;.$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad$ etc.

Variances $\quad s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2 \quad$ etc.

s_x is standard deviation

Covariance $\quad \mathrm{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $\quad r = \frac{\mathrm{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \qquad b = \frac{\mathrm{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.