The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Further Civil Engineering Mathematics and Computing **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions; if you do, only your best four will be counted.*

1 (i) Use the properties of trigonometric functions to prove the identity

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } m = n \neq 0. \\ 2L, & \text{if } m = n = 0. \end{cases}$$

(12 marks)

(ii) If air resistance is proportional to the square of the instantaneous velocity, then the velocity $v(t)$ of a mass m dropped from a given height is determined by the differential equation

$$m\frac{dv}{dt} = mg - kv^2$$

where $g = 9.8$ m/s^2 is the gravitational acceleration and $k = 0.125$ kg/m is the air resistance coefficient. Let $v(0) = 0$ and $m = 0.5$ kg. Use the fourth-order Runge-Kutta method with $h = 0.1$ to approximate the velocity $v(0.2)$. Work correct to *four* decimal places. ***(13 marks)***

2 (i) The power of an engine depends on two parameters, x and y, through the formula

$$P = \sqrt{\frac{x^3}{y}}.$$

Under ideal working conditions, $x = 28$ and $y = 10$ (in arbitrary units). The quantity x is subjected to an increase of 10 %. Use the small error formula to calculate the approximate percentage change needed in y to ensure that the power of the engine does not change. ***(10 marks)***

(ii) The 4th order Runge-Kutta method applied to a differential equation allows the estimation of the unknown function $y(x)$ at $x = 1$ with two different step-lengths as

h	$y(1)$
0.1	0.751140
0.2	0.751125

Use this data to estimate a value for h that will ensure an accuracy of 6 decimal places in the value of $y(1)$ calculated by the 4th order Runge-Kutta method. ***(7 marks)***

(iii) Explain why the equation

$$3x - \cos x - 1 = 0$$

has a root in the interval $(0.55; 0.75)$. Perform *four* iterations of the bisection method for finding this root. Work correct to three decimal places. ***(8 marks)***

3 (i) If $w = 1/r$ and $r = x^2 + y^2 + z^2$, use the chain rule to calculate

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}.$$

(6 marks)

(ii) Find the first three non-zero terms of the series solution of the differential equation

$$y'' - \frac{4x}{x^2+1}y' + \frac{6}{x^2+1}y = 0, \quad y = y(x),$$

subject to the conditions $y(1) = 1$ and $y'(1) = 0$. ***(10 marks)***

(iii) The finite difference form of the first and second order derivative of a function, $U(x,t)$, can be written as

$$\left(\frac{\partial U}{\partial t}\right)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k}, \qquad \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2},$$

where $U_{i,j} = U(ih, jk)$.

(a) Show that the *explicit* approximation to the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 5U, \quad U = U(x,t), \tag{1}$$

is given in the usual notation as

$$U_{i,j+1} = rU_{i-1,j} + (1 - 2r - 5k)U_{i,j} + rU_{i+1,j},$$

where $r = k/h^2$ and k and h denote the time and spatial steps, respectively.

(b) If the initial condition and boundary conditions associated with the equation (1) are

$$U(x,0) = 1 + x(1-x), \quad 0 \le x \le 1,$$
$$U(0,t) = U(1,t) = 0, \quad t \ge 0,$$

use the explicit scheme with $h = 0.2$ and $k = 0.002$ to find the values of $U(x,t)$ at $t = 0.002$ working correct to three decimal places.

(9 marks)

4 (i) Find the Fourier series decomposition of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

(15 marks)

(ii) Use the above series at $x = \pi/2$ to show the property

$$\frac{\pi^3}{16} = \sum_{k=0}^{\infty}\left[\frac{6}{(2k+1)^3} - \frac{\pi^2}{2k+1}\right].$$

(10 marks)

5 (i) Use the method of separation of variables to show that there is an oscillating solution of the equation

$$\frac{\partial^2\Phi}{\partial x^2} = \Phi + \frac{\partial\Phi}{\partial y}.$$

Show that the general solution which satisfies the boundary conditions $\Phi(0, y) = \Phi(\pi, y) = 0$ may be written as

$$\Phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-(1+n^2)y} \sin nx,$$

where B_n are constants. Find also the particular solutions which satisfy the boundary condition

$$\Phi(x, 0) = x^2, \qquad 0 \leq x \leq \pi.$$

(20 marks)

(ii) Show that the function

$$G(x, y) = 2y^2 \exp\left[-\frac{x^2 + y^2}{2}\right],$$

is a solution of the differential equation

$$y\frac{\partial G}{\partial x} - x\frac{\partial G}{\partial y} + \frac{2x}{y}G = 0.$$

(5 marks)

End of Question Paper

Formula sheet

- **The fourth-order Runge-Kutta method**

  For a first order initial value problem

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

  we have

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h/2, y_n + k_1/2)$$
$$k_3 = hf(x_n + h/2, y_n + k_2/2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
$$y_{n+1} = y_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

  with $n = 0, 1, 2, \ldots$

- The local truncation error in the case of the 4th order Runge-Kutta method is given by

$$Y(x) - y(x) = Ch^4$$

  where Y(x) is the estimated value, y(x) is the exact value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

  If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

  If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

  If the function $f(x)$ is defined over the interval $-l \leq x \leq l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \le x \le l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

- **Some trigonometric identities**

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1, \quad \sin 2\alpha = 2\sin\alpha\cos\alpha$$