The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011-2012**

Mathematics for Engineering Modelling **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Derive the Maclaurin series for e^x and $\ln(1+x)$, each up to the x^3 term.

Find the first two non-zero terms in the Taylor series for $\sin(x)$ about the point $x = \pi$.

(10 marks)

(ii) Given the infinite geometric series

$$a + a\,r + a\,r^2 + \ldots + a\,r^{n-1} + \ldots, \qquad r \neq 1,$$

derive an expression for the partial sum s_n of the first n terms. Hence show that the series is convergent for $|r| < 1$ and find the sum of the infinite series.

Find the sum of the infinite series

$$1 + 8\,x^3 + 64\,x^6 + 512\,x^9 + \ldots,$$

and determine its radius of convergence, R. Hence find, valid for $|x| < R$, the sum of the infinite series

$$24\,x^2 + 384\,x^5 + 4608\,x^8 + \ldots$$

(10 marks)

(iii) Evaluate

$$\lim_{x \to 0} \frac{\ln(x+1) - x}{x\,\sin x}.$$

(5 marks)

2 Consider the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases},$$

and its Fourier series denoted by $F(x)$.

(i) Sketch $F(x)$ in the range $-3\pi < x < 3\pi$ and determine the value of $F(0)$.

(6 marks)

(ii) Show that

$$F(x) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi} \sin(2m+1)x.$$

(14 marks)

(iii) Using a suitable choice for x, deduce that

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}.$$

(5 marks)

3 (i) Find *by integration* the Laplace transform $F(s)$ of $f(t) = t\,\mathrm{e}^{at}$.

(3 marks)

(ii) With the aid of the table of Laplace transforms, find

(a) the Laplace transform of

$$\mathrm{e}^{2t}\,\sin 3t;$$

(b) the inverse transform of

$$\frac{1}{(s-1)(s+2)}.$$

(6 marks)

(iii) Use the method of Laplace transforms to solve the ordinary differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4\,\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = 0,$$

subject to the initial conditions $y = 1$ and $\mathrm{d}y/\mathrm{d}t = 2$ at $t = 0$.

(9 marks)

(iv) The variables $y_1(t)$ and $y_2(t)$ satisfy the coupled system of ordinary differential equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2\,,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = -6\,y_1 + 5\,y_2\,,$$

subject to the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$. Using Laplace transforms, solve for $y_1(t)$. Hence find $y_2(t)$ without using its Laplace transform.

(7 marks)

4 The vibration of a string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The boundary conditions are that $u = 0$ at both $x = 0$ and $x = a$, and initially, i.e. at $t = 0$, $\partial u/\partial t = 0$.

(i) Using the *method of separation of variables*, show that this configuration has the general solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{a}x \cos\frac{n\pi}{a}ct.$$

(16 marks)

(ii) If initially $u = f(x)$, find an expression for A_n in terms of $f(x)$.

(6 marks)

(iii) Show that each term the general solution can be expressed in terms of

$$\sin\left[\frac{n\pi}{a}(x+ct)\right] \quad \text{and} \quad \sin\left[\frac{n\pi}{a}(x-ct)\right],$$

and give a brief physical interpretation.

(3 marks)

5 (i) Let R be the rectangular region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = \ln 2$. Evaluate the integral

$$\int\int_R y\,\mathrm{e}^{xy}\,\mathrm{d}x\,\mathrm{d}y\,.$$

(6 marks)

(ii) By changing the order of integration, evaluate

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y\,.$$

(9 marks)

(iii) A region R is given by the bounds $x^2 + y^2 \leq 1$, $x \geq 0$ and $y \geq 0$. Evaluate, using a change of coordinates, the integral

$$\int\int_R 2\sqrt{1 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y\,.$$

(10 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P\ \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1-r^n)/(1-r),\ \ r \neq 1\ .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a=0$:

$$f(x) = f(0) + f'(0)\ x + \frac{1}{2!}f''(0)\ x^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!} x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with the radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx \quad ,$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx\,, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx\,, \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt \quad .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then
$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \quad .$$
2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then
$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} \quad .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(x,y)$ about $x = a, y = b$ is

$$\begin{aligned} f(x,y) &= f(a,b) + \{(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\} + \\ &+ \frac{1}{2!}\Big\{(x-a)^2\, f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + \\ &+ (y-b)^2\, f_{yy}(a,b)\Big\} + \\ &+ \ldots \end{aligned}$$

Here $f_x = \dfrac{\partial f}{\partial x}$ etc.

Partial Differential Equations **(2 independent variables)**

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn.

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos \omega x + B\sin \omega x$$
$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh \omega x + D\sinh \omega x$$
$$\text{or } Ee^{\omega x} + Fe^{-\omega x}$$
$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\ (n \geq 1)$	$1/s^n$
e^{at}	$\frac{1}{s-a}$
$\frac{1}{a}\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)\ f(t-a)$	$e^{-as} F(s)$, where $F(s) = L(f(t))$