The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Mathematics (Computational and Numerical Methods) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Explain why the equation

$$3x - \cos x - 1 = 0 \qquad (1)$$

has a root in the interval $(0.55; 0.75)$. Perform *four* iterations of the bisection method for finding his root. Work correct to three decimal places. ***(8 marks)***

(ii) When applying the bisection method for finding the root of the equation (1), the error, ϵ, is bounded using the formula

$$\frac{1}{2^n}(b-a) \le \epsilon,$$

where a and b are the two starting values. Estimate the number of iterations needed to find a root with a precision of 10^{-5}. ***(4 marks)***

(iii) Given the matrix

$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix},$$

perform three iterations with the power method in order to establish the largest eigenvalue starting from the initial guess $[1, 1, 1]^T$. Work throughout with three decimal places. ***(13 marks)***

2 (i) The Lagrange interpolation polynomial of least degree which passes through the $(n+1)$ points (x_i, f_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f_i,$$

where

$$L_i = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)},$$

and $f_i = f(x_i)$. Given that the x_i values are tabulated at a constant interval h and using the variable change $x = x_0 + rh$, show that the Lagrange polynomial passing through 3 points (x_i, f_i), $i = 0, 1, 2$ can be expressed as

$$P_2(r) = \frac{1}{2}[(r-1)(r-2)f_0 - 2r(r-2)f_1 + r(r-1)f_2].$$

Hence derive the approximation

$$P_2'(x_1) \approx (f_2 - f_0)/(2h).$$

(10 marks)

(ii) Given the table

x	1	1.5	2	2.5	3
$f(x)$	1		2.5198		4.3267
$f'(x)$	1.3333	1.5263		1.8096	1.9230

use the differentiation formula given in part (i) to calculate $f'(2)$. Hence, use the above data with the composite Simpson's rule to estimate

$$I = \int_1^3 f'(x)dx.$$

Work correct to *four* decimal places. ***(6 marks)***

(iii) A linear programming problem is defined as

- Variables: x_1 and x_2
- Constraints:

$$x_1 + 3x_2 \geq 9,$$
$$-x_1 + x_2 \leq 5,$$
$$2x_1 + 3x_2 \leq 24,$$
$$x_1 - 3x_2 \leq 6,$$
$$x_1 \geq 0, \quad x_2 \geq 0,$$

- Objective function:

$$f = 4x_1 + 5x_2.$$

Plot the constraints, determine the feasible region on a diagram and solve the problem so that the objective function is maximum. Calculate the equation of the iso-profit line and draw it on the diagram. ***(9 marks)***

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3 (i) It is suspected that the data

x	0.1	0.2	0.3	0.4	0.5
y	0.52	0.68	0.83	1.04	1.3

can be represented by $y = ab^{2x}$, where a and b are constants. By employing a suitable transformation and then using the least square linear fit, find the least square values of a and b correct to two decimal places. ***(15 marks)***

(ii) Find the first three non-zero terms of the series solution of the differential equation

$$y'' - \frac{4x}{x^2+1}y' + \frac{6}{x^2+1}y = 0, \quad y = y(x), \tag{2}$$

subject to the conditions $y(1) = 1$ and $y'(1) = 0$. ***(10 marks)***

4 Find the solution of the differential equation at $x = 3/4$

$$y'' + xy' - x = 0,$$

using finite central difference approach given that $y(0) = 0$, $y\prime(1) = 1$ with $0 \leq x \leq 1$ and $h = 1/4$. Solve the resulting system of equation using exact arithmetics with an accuracy of three decimal places. ***(25 marks)***

End of Question Paper

Formula sheet

- **The fourth-order Runge-Kutta method**

  For a first order initial value problem

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

  we have

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h/2, y_n + k_1/2)$$
$$k_3 = hf(x_n + h/2, y_n + k_2/2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
$$y_{n+1} = y_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

  with $n = 0, 1, 2, \ldots$

- The local truncation error in the case of the 4th order Runge-Kutta method is given by

$$Y(x) - y(x) = Ch^4$$

  where Y(x) is the estimated value, y(x) is the exact value, C is a constant and h is the step size used in the numerical scheme.

- Assuming that the x_i values are free of errors, the normal equations used in the process of interpolation in the least-square sense are

$$\sum_{j=0}^{n} a_j \sum_{i=0}^{n} x_i^{j+k} = \sum_{i=0}^{n} x_i^k f_i, \quad k = 0, 1, 2. \ldots$$

- The first and second order derivatives can be written as finite differences as

$$U'_k = \frac{U_{k+1} - U_k}{h}, \quad \text{forward formula}$$
$$U'_k = \frac{U_k - U_{k-1}}{h}, \quad \text{backward formula}$$
$$U'_k = \frac{U_{k+1} - U_{k-1}}{2h}, \quad \text{central formula}$$
$$U''_k = \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2},$$

  where U_k and $U_{k\pm1}$ denote the values of $U(x_k)$ and $U(x_k \pm h)$, respectively.