The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-2012**

Vectors and Fluids **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Consider the vector field $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$, where Ω is a constant. Calculate

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}, \quad \boldsymbol{\nabla} \times \boldsymbol{u}, \quad \boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}), \quad (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u},$$

and verify that the following result holds for this case:

$$\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u}.$$

(11 marks)

Sketch the vector field and describe the motion in no more than a few words.

(3 marks)

(ii) Let

$$\phi = \ln r\,, \qquad \text{where} \qquad r = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{and} \qquad \boldsymbol{\nabla}\phi = \frac{1}{r^2}\,\boldsymbol{r}.$$

(6 marks)

Hence calculate the rate of change of ϕ in the direction $(1, 2, 0)$ at the point $(a, 0, 0)$.

(5 marks)

2 (i) Find an expression for the i^{th}-component of the result $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$ in suffix notation.

Hence use the suffix notation to show that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi) = 0.$$

(6 marks)

Show that $\boldsymbol{\nabla} \times \boldsymbol{u} = 2\,\boldsymbol{\Omega}$, where $\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$, using the result

$$\epsilon_{ijk}\,\epsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\,.$$

(8 marks)

(ii) State the conditions on a 3×3 matrix, L, for it to be a valid matrix of transformation, representing a rotation of frames about a common origin.

(2 marks)

Let L be a valid matrix of transformation of the form

$$L = \alpha \begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Use the determinant of L to calculate the value of α.

Find μ and κ such that the vector $\boldsymbol{\lambda} = (\mu, \kappa, 1)$ is aligned with the axis of rotation between frames.

Using the vector $\boldsymbol{v} = (1, 0, 0)$, perpendicular to $\boldsymbol{\lambda}$, calculate the angle of rotation between frames.

(9 marks)

3 The cylindrical polar coordinate system is related to the Cartesian system by

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad z = z.$$

(i) Find expressions for δx and δy in terms of r, θ, δr and $\delta\theta$.

(5 marks)

Use your results to find an expression for

$$\delta l^2 = \delta x^2 + \delta y^2 + \delta z^2,$$

in terms of r, θ, δr, $\delta\theta$ and δz, and state the property of this expression that implies that the cylindrical polar coordinate system is orthogonal. Find expressions for h_1, h_2 and h_3 (in standard notation) in terms of r and θ.

(6 marks)

(ii) The vector field $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = (x^2 + y^2)z\,\boldsymbol{i} + z^3\,\boldsymbol{k}.$$

Sketch the volume V bounded by $r = a$ and planes $z = 0$ and $z = h$, and verify that Gauss's Theorem, namely

$$\int_S \boldsymbol{F}\cdot \mathbf{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V,$$

holds for the given V and $\boldsymbol{F}$.

(14 marks)

4 The flow $\boldsymbol{u}$ of an incompressible fluid of density ρ is given by the velocity potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - 2aU\,\theta\,,$$

in cylindrical polar coordinates (r, θ, z), around a solid of radius a.

(i) State why ϕ must satisfy the Laplace equation.

(2 marks)

(ii) Given that

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}},$$

find the velocity field $\boldsymbol{u} = \boldsymbol{\nabla}\phi$.

(3 marks)

(iii) Verify that the appropriate boundary conditions are satisfied on $r = a$, and that the flow far from the origin is $U\boldsymbol{i}$.

(5 marks)

(iv) Find the only stagnation point.

(7 marks)

(v) Using your results, sketch the flow.

(4 marks)

(vi) Neglecting gravity and assuming that p tends to the constant value p_∞ as $r \to \infty$, deduce that the maximum value of p is $p_\infty + \rho U^2/2$.

(4 marks)

5 The flow of a fluid has velocity field

$$\boldsymbol{u} \;=\; u_1(x, y, z)\,\boldsymbol{i} + (\sigma x - 2\alpha y + \beta z)\,\boldsymbol{j} + (\gamma x + \beta y + \alpha z)\,\boldsymbol{k},$$

where α, β, γ and σ are constants. The flow is irrotational and has a stagnation point at the origin. The fluid is incompressible. Use these properties to show that

$$u_1(x, y, z) = \alpha x + \sigma y + \gamma z.$$

(12 marks)

Deduce that there must exist a velocity potential ϕ for $\boldsymbol{u}$, and find a ϕ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$.

(7 marks)

If $\alpha = \gamma = \sigma = 0$, find a stream function $\psi = \psi(y, z)$ such that $\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi\boldsymbol{i})$.

(6 marks)

End of Question Paper