The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2011-2012**

Applied Differential Equations **2 hours**

Attempt all FOUR questions.

1 For the equation $y'(x) = f(x, y(x))$, the AB3 method is defined as

$$y_{n+1} = y_n + \frac{1}{12}h(23f_n - 16f_{n-1} + 5f_{n-2}),$$

and the AM2 method is defined as

$$y_{n+1} = y_n + \frac{1}{12}h(5f_{n+1} + 8f_n - f_{n-1}).$$

For the AB3 method, the local truncation error is $T^P = \frac{3}{8}h^4 y^{(4)}(\xi_1)$, while for AM2 it is $T^C = -\frac{1}{24}h^4 y^{(4)}(\xi_2)$, where $x_{n-2} \le \xi_1 \le x_{n+1}$ and $x_{n-1} \le \xi_2 \le x_{n+1}$.

(i) Given the differential equation and initial condition

$$y'(x) = 3y\sin(x), \quad y(0) = 1, \tag{1}$$

and the values $y(0.1) = 1.0151$ and $y(0.2) = 1.0616$, apply the ABM method (with AB3 as the predictor and AM2 as the corrector) to find the approximate solution at $x = 0.3$, using step size $h = 0.1$. Work throughout correct to four decimal places (Note the argument of $\sin(x)$ should be understood in radians). ***(8 marks)***

(ii) Estimate the local truncation error for the approximate solution at $x = 0.3$ using the Milne's device. ***(10 marks)***

(iii) Show that the AB3 method is convergent. ***(7 marks)***

2 A single step method is defined by the following formulae:

$$k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}k_1\right),$$
$$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{3}{4}k_2 \; .$$

(i) Write down the local discretization error of the method. Show that the method is consistent. ***(9 marks)***

(ii) Find the interval of absolute stability for the method when it is applied to the test equation $y' = \lambda y$. ***(10 marks)***

(iii) The following table contains the grid-point values of the two solutions, $Y_1(x)$ and $Y_2(x)$, of a linear differential equation $y''(x) = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method.

x	0.5	1.0	1.5	2.0
$Y_1(x)$	0.8047	0.8969	1.5839	3.5081
$Y_2(x)$	1.3202	2.1468	4.4165	10.1376

$Y_1(x)$ was determined using the initial conditions $y(0) = 1$, $y'(0) = 0.5$, and $Y_2(x)$ was obtained using $y(0) = 1$, $y'(0) = -0.5$. Form a linear combination of these two solutions which is the numerical solution to the equation $y''(x) = f(x, y, y')$ with the boundary conditions

$$y(0) = 1, \quad y(2) = 2.$$

Calculate the solution $y(x)$ at each x-value given in the table. ***(6 marks)***

3 The function $u(x,t)$ satisfies the wave equation in a finite domain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 \le x \le \ell, t \ge 0), \tag{2}$$

with homogeneous boundary conditions $u(0,t) = u(\ell,t) = 0$, and initial conditions

$$u(x,0) = 0, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 2\sin(\pi x/\ell) \ ,$$

where $\ell > 0$ is a constant.

(i) For a separable solution of the form $T(t)X(x)$, show that $X(x)$ and $T(t)$ are given by

$$X''(x) - \alpha X(x) = 0, \quad T''(t) - c^2\alpha T(t) = 0,$$

where α is a constant. Show that for non-trivial solutions

$$X(0) = X(\ell) = 0, \quad T(0) = 0 \ .$$

(8 marks)

(ii) Assuming $\alpha = -s^2$ $(s > 0)$, find the values of s such that $X(x)$ and $T(t)$ have non-trivial solutions, and thus find $X(x)$ and $T(t)$. ***(12 marks)***

(iii) The general solution of equation (2), with the given boundary conditions, can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \sin\frac{n\pi c t}{\ell} \ ,$$

where b_n are constants to be determined. Find the values of b_n such that $u(x,t)$ satisfies the given initial conditions, hence write down the solution for $u(x,t)$. ***(5 marks)***

4 (i) The function $w(x,y)$ satisfies the Laplace's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

in the rectangular region $0 \le x \le a\,, 0 \le y \le b\,$, subject to boundary conditions:

$$w(0,y) = f(y), \quad w(a,y) = g(y), \quad w(x,0) = F(x), \quad w(x,b) = G(x).$$

The function $w(x,y)$ can be found as the sum of two functions $u(x,y)$ and $v(x,y)$ where $u(x,y)$ and $v(x,y)$ both satisfy the Laplace's equation with suitable boundary conditions. Find the boundary conditions for u and v so that u and v can be determined using the usual method of separation of variables. Do **NOT** attempt to solve the equations.

(4 marks)

(ii) The function $u(x,y)$ satisfies a second order hyperbolic partial differential equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0,$$

where a, b and c are constants, and a and c are nonzero. Let us define the new variables η and ν as

$$\eta = x + py, \quad \nu = x + qy,$$

where p and q are two unknown constants and $p \ne q$. Find the values of p and q in terms of a, b and c, such that, in the new variables, the equation becomes

$$\frac{\partial^2 u}{\partial \eta \partial \nu} = 0.$$

When $a = 2$, $b = 3$, and $c = 1$, write down the general solution of the equation as a function of x and y. ***(21 marks)***

End of Question Paper