Data provided: "Statistics Tables" by H.R. Neave

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MAS275

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–21; Q2–10; Q3–26; Q4–20; Q5–23)

1 A (delayed) renewal process is defined by rolling a six-sided die repeatedly, and saying that a renewal occurs whenever a run of three consecutive identical rolls is completed. You should assume that the rolls are independent of each other and that the die is fair.

(a) Let v_n be the probability that a renewal occurs at time n. Explain why $v_1 = v_2 = 0$, and give the value of v_n for $n \geq 3$. Hence show that the generating function $V(s)$, defined as $\sum_{n=0}^{\infty} v_n s^n$ for $|s| < 1$, has the form

$$V(s) = \frac{s^3}{36(1-s)}.$$

(8 marks)

(b) Let u_n be the probability that, given that a renewal occurs at time t, a renewal occurs at time $t+n$. Explain why $u_1 = 1/6$ and $u_2 = 1/36$, and give the value of u_n for $n \geq 3$. Hence show that the generating function $U(s)$, defined as $\sum_{n=0}^{\infty} u_n s^n$ for $|s| < 1$, has the form

$$U(s) = 1 + \frac{1}{6}s + \frac{s^2}{36(1-s)}.$$

(8 marks)

(c) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of rolls until the first renewal.

(5 marks)

2 A Markov chain has state space $\{1, 2, 3, 4, 5\}$. There are three communicating classes, which are $\{1, 2\}$, $\{3, 4\}$ and $\{5\}$. The class $\{1, 2\}$ is persistent and aperiodic, the class $\{3, 4\}$ is persistent and has period 2, and the class $\{5\}$ is transient. Give a possible transition matrix for this Markov chain. ***(10 marks)***

3 Let (X_n) and (Y_n) be Markov chains on the state space $\{1, 2, 3, 4\}$ with transition matrices

$$P_X = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

and

$$P_Y = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

respectively. Each chain starts in state 1 at time 0.

(a) Find the unique stationary distribution of each chain. ***(10 marks)***

(b) Show that, for each $i \in \{1, 2, 3, 4\}$, $P(X_n = i)$ converges to a limit as $n \to \infty$, and give the value of the limit in each case. (You may use results from the course.) ***(10 marks)***

(c) Does $P(Y_n = 1)$ converge to a limit as $n \to \infty$? Give a reason for your answer. ***(6 marks)***

4 A sports league is divided into four divisions, labelled 1, 2, 3 and 4. At the end of each year, if team A is in division n for $n = 1$, 2 or 3 they will be relegated (moved to division $n + 1$ for the next year) with probability $1/8$, and if team A is in division n for $n = 2$, 3 or 4 they will be promoted (moved to division $n - 1$ for the next year) with probability $1/8$. Otherwise they will remain in the same division for the next year. Let A_n be the division that team A is in in the year n of the league, and assume that the random variables (A_n) for $n \geq 0$ can be modelled as a Markov chain on the state space $\{1, 2, 3, 4\}$.

(a) Give the transition matrix of the Markov chain. ***(4 marks)***

(b) Given that team A starts in Division 4 in year 0, find the expected number of years until they are in Division 1. ***(8 marks)***

(c) Given that team A starts in Division 2 in year 0, find the probability that they are in Division 1 before they are in Division 4. ***(8 marks)***

5 A shop is open for 8 hours a day. Treating the shop's day as the interval $(0,8]$ with time measured in hours, customers arrive at the shop according to a variable rate Poisson process with rate $\lambda t = 24t - 3t^2$ for $t \in (0,8]$.

(a) What is the distribution of the number of customers who arrive in the shop in the day? ***(4 marks)***

(b) Use a Normal approximation to find the approximate probability that the number of customers who arrive in the day is at least 280. ***(5 marks)***

(c) Find the exact probability that at most 10 customers arrive in the first hour of the day. ***(7 marks)***

(d) Given that exactly 8 customers arrive in the first hour, find the conditional probability that none of them arrive in the first half-hour. ***(7 marks)***

End of Question Paper